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A Theory of Threshold Contracts*

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We consider an infinitely repeated reappointment game in a principal–agent relationship. Typical examples are voter–politician or government–public servant relationships. The agent chooses costly effort and enjoys being in office until he is deselected. The principal observes a noisy signal of the agent's effort and decides whether to reappoint the agent or not. We analyse the stationary Markovian equilibria of this game and examine the consequences of threshold contracts, which forbid reappointment if the principal's utility is too low. We identify the circumstances under which such threshold contracts are welfare-improving or beneficial for the principal.

Keywords: principal–agent model, repeated game, reappointment, stationary Markovian strategies, threshold strategies, threshold contracts, asymmetric information, commitment.

JEL Classification: C83, D82, D86, H11

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1. Introduction

The present paper is motivated by two aspects of a reappointment setting—reappointment as an incentive to exert effort on the one hand, and the consequences of a threshold contract on the other.

Typical examples of reappointment decisions in a principal–agent relationship are voters who reelect or deselect a member of the executive or legislative branch in a democracy. Similar reappointment situations occur when a manager decides whether to extend an employee's contract or not. Prominent reappointment decisions are also taken by governments. The House of Representatives and the Senate in the United States, for instance, decide whether to reappoint the President and the board members of the Federal Reserve.

A common characteristic of such situations is that the reappointment decision is the sole incentive device available. This is most obvious in a democracy, but there are many other cases in which the principal deciding on the renewal of an agent's contract is confronted with a fixed wage scheme and thus cannot resort to monetary incentive contracts. In such reappointment situations, the principal would like to motivate the office holder to exert high effort by threatening to deselect him otherwise. When, however, reappointment decisions are taken repeatedly, and new candidates for office are likely to behave like the preceding incumbents, the threat may turn out to be ineffectual. To what extent a principal can motivate an agent to work hard by reappointment decisions in a potentially infinitely repeated game is the first issue this paper addresses.

A recent strand of literature summarized in Gersbach (2012) has suggested that threshold contracts might increase the utility of voters, who are the principal in a democracy. In such a contract, a candidate for office commits to a freely chosen particular value that he will deliver to voters. If he fails to generate this threshold utility for the voters, the office holder cannot stand for reelection, i. e. he is automatically deselected. If he meets this self-imposed threshold, voters are free to reelect or deselect him. It is unknown whether and how reelection threshold contracts affect the utility of voters when repeated elections with threshold contracts are considered. This is the second issue we deal with in this paper.

A key element of the threshold contracts we consider is that they are only binding with respect to deselection, i. e. they forbid reappointment if the threshold has been missed, but do not enforce reappointment if the threshold has been reached. This is a necessary feature of a democratic process, where each appointment decision must be legitimized by an election. Yet one can think of several other situations in which the principal does not want to commit to the agent's reappointment, or is not able to.

We study an infinitely repeated reappointment game played by a principal and an agent. In each period, the agent chooses costly effort and enjoys a fixed benefit from holding office. At the end of each period, the principal observes his payoff, which is, in expectation, monotonically increasing in the agent's effort, but is affected by random events. The principal decides whether to reappoint the agent or not. If the agent is reappointed, the game continues. Otherwise it ends. We first analyse the stationary Markovian equilibria (henceforth "equilibria") of this game. Second, we allow the agent to write threshold contracts. The main results of the first part are:

1. We characterize equilibria by necessary and sufficient conditions.

2. We show how equilibria can be constructed.

3. We illustrate how equilibria can be determined for the case of uniform noise distribution.

In the second part of the paper, we allow the agent to write threshold contracts from the outset. We restrict the analysis to threshold strategies; that is, the principal reappoints the agent if and only if the principal's payoff is above a particular threshold. The main results of the second part are:

4. We characterize equilibria with threshold contracts.

5. We establish conditions under which threshold contracts are welfare-improving, or beneficial from each of the players' point of view.

6. We illustrate our findings by examples.

7. Finally, we outline various ways to embed the game into larger games, and we discuss how to eliminate equilibria in which the agent's effort is low.

Relation to the literature

Barro (1973), Ferejohn (1986), and Austen-Smith and Banks (1989) have examined repeated elections in the presence of moral hazard. Acting as the principal, voters try to motivate office-holders to work hard. Reappointment decisions, rather than compensation contracts[1], are the only motivational tool available to the principal. Radner (1986) explores the incentive effects of different strategies to review the agent's performance once every k periods and to deselect the agent if the average performance is below some cut-off level.

In Ferejohn (1986) and Austen-Smith and Banks (1989), the voters choose a reelection rule to motivate incumbents to take costly actions. The threat of deselection enables the voters to rein the politicians back. The retrospective voting rule[2] used by the voters gives the incumbents maximal incentive. The principal's reappointment decision must be an equilibrium response after he has observed the payoff. Our basic set-up is quite similar, yet we focus on equilibria involving simple cut-off rules: the current incumbent is reelected as long as the utility of the voters is above a certain threshold.

In its quest for equilibrium reappointment responses, our paper is closest—and complementary—to the seminal work of Banks and Sundaram (1993). They allow for both adverse selection and moral hazard in a model with infinite time horizon. In a second paper, Banks and Sundaram (1998) identify equilibrium retention rules when the agent faces a limit of two terms in office.

In contrast to Banks and Sundaram (1993), we focus on stationary Markovian strategies, rather than trigger strategies. Further, we consider the consequences arising when the agent can use a threshold contract to commit to a threshold performance level he will have to achieve in order to stand for reappointment. We explore the circumstances under which cut-off rules provide high incentives for incumbents and identify cases when threshold contracts are necessary to generate favourable outcomes.

Our work provides a foundation for threshold contracts as discussed in Gersbach (2012) and examined in a one-shot game in Gersbach and Liessem (2008). The consequences of the repeated application of threshold contracts have not yet been explored. This is the topic of the present paper.

[1] There is a much larger body of literature on compensation contracts (see Bolton and Dewatripont, 2005).

[2] See Fiorina (1981) for an in-depth discussion of such voting rules.

Structure of the paper

Our paper is organized as follows: In Section 2 we describe the game and introduce the equilibrium concept we are going to use. In Section 3 we introduce the concept of stationary Markovian strategy and stationary Markovian equilibrium. We then determine the stationary Markovian equilibria of the reappointment game. We illustrate our findings for the example of uniformly distributed noise. In Section 4 we analyse the effects of threshold contracts prohibiting reappointment if the principal's utility is below the threshold defined in the contract. Section 5 contains several examples illustrating our findings. In Section 6 we discuss variants of the model. In particular, we extend our model by the principal's initial decision on whether to appoint the agent or not and examine the consequences of this extension. Section 7 concludes.

2. The Reappointment Game

We consider a reappointment game with a potentially infinite number of periods, played by a principal ("voter") and an agent ("politician"). Time is indexed by $t = 1, 2, \ldots$ The agent is already in office at the beginning of period 1. At the end of each period, the principal decides whether to renew the agent's contract for the next period. We call this *reappointment*. If the agent is reappointed, he stays in office and the game continues for (at least) one more period; if he is not reappointed, the reappointment game ends.

As long as the agent is in office, he chooses a level of effort $e_t \in [\underline{e}; e^{\sup}) \subseteq \mathbb{R}$ with $\underline{e} < e^{\sup} \leq \infty$ at the beginning of each period t. From this effort level e_t, the agent derives utility $v(e_t)$ in period t. The utility function $v\colon [\underline{e}; e^{\sup}) \to \mathbb{R}$ is assumed to be strictly decreasing and strictly concave with $\lim_{e \to e^{\sup}} v(e) = -\infty$. Thus the agent faces increasing disutility as well as increasing marginal disutility of effort, and circumstances become unbearable when effort approaches its theoretical maximum.

The agent discounts future utility with a per-period discount factor of $\beta \in (0; 1)$. If he is not in office, his fixed per-period utility will be v_*, which thus is his outside option. In order to rule out trivial cases, we assume $v(\underline{e}) > v_*$. This means that to his outside option the agent will prefer the situation where he is in office exerting the least possible effort. The agent is also assumed to be risk-neutral.

For $t \in \mathbb{N}^*$ and $\tau \in \mathbb{N}_0$, let $r_{t,\tau}$ denote the subjective probability with which, at the beginning of period t, the agent believes that he will still be in office in period $t+\tau$. As the agent can only be recalled at the end of a period, we have $r_{t,0} = 1$. The agent's intertemporal expected utility at the beginning of period t is given by

$$\mathfrak{v}_t := \sum_{\tau=0}^{\infty} r_{t,\tau} \beta^\tau v(e_{t+\tau}) + \sum_{\tau=1}^{\infty} (1 - r_{t,\tau}) \beta^\tau v_* \,,$$

which can be rewritten as

$$\mathfrak{v}_t = (1-\beta)^{-1} v_* + \sum_{\tau=0}^{\infty} r_{t,\tau} \beta^\tau \big(v(e_{t+\tau}) - v_* \big). \tag{1}$$

When the agent is in office in period t and his effort is e_t, the principal's utility in period t shall be given by

$$w_t := e_t + a_t,$$

where a_t is a realization of a random variable A_t representing stochastic fluctuations of the environment beyond the control of the decision-maker. These are called "noise". The random variables A_t are assumed to be stochastically independent across time and identically distributed with $\mathrm{E}[A_t] = 0$ for all t. We use the capital letter A_t for the random variables and the corresponding small letter a_t for their realizations. Further, we drop the index t in connection with the random variable A when the period does not matter. The principal is risk-neutral and discounts future utility with a per-period factor of $\gamma \in (0;1)$. We use w_* to denote the principal's per-period expected utility when the agent is not in office. Thus, w_* is the principal's outside option.

Analogously to $r_{t,\tau}$, we use $s_{t,\tau}$ to denote the principal's period-t belief that the agent will be in office in period $t+\tau$. Trivially, $s_{t,0} = 1$. Since $\mathrm{E}[A] = 0$, the principal's expected utility at the beginning of period t is given by

$$\mathfrak{w}_t := \sum_{\tau=0}^{\infty} s_{t,\tau} \gamma^\tau e_{t+\tau} + \sum_{\tau=1}^{\infty} (1 - s_{t,\tau}) \gamma^\tau w_*,$$

which can be rewritten as

$$\mathfrak{w}_t = (1-\gamma)^{-1} w_* + \sum_{\tau=0}^{\infty} s_{t,\tau} \gamma^\tau (e_{t+\tau} - w_*) \,. \tag{2}$$

In each period, the agent has to choose his effort level e_t before the noise level A_t is realized. The principal observes the realized utility level w_t before deciding on reappointment, but neither the effort level e_t nor the noise level a_t are imparted to him. The chronological sequence in each period is as follows:

1. The agent chooses his effort level e_t, which is private information and cannot be observed by the principal.

2. Principal and agent observe the principal's utility level w_t.

3. The principal decides whether to reappoint the agent or not.

Formally, every pure strategy of the agent can be described by an infinite sequence $\mathbf{e} = (e_1, e_2, \dots)$ of measurable functions $e_t \colon \mathbb{R}^{t-1} \to [\underline{e}; e^{\sup})$. The value $e_t(w_1, \dots, w_{t-1})$ is the effort level chosen by the agent in period t if he is still in office in that period and utility levels of $w_1, \dots, w_{t-1}$ have been realized in the previous periods. The first function e_1 is a constant. Analogously, every pure strategy of the principal is given by a sequence $\mathbf{p} = (p_1, p_2, \dots)$, with $p_t \colon \mathbb{R}^t \to \{0, 1\}$ being measurable functions. The value $p_t(w_1, \dots, w_t) = 1$ indicates that the principal reappoints the agent at the end of period t after having observed utility levels of $w_1, \dots, w_t$.

For $\tau \geq 1$, we define $\delta_{t,\tau}$ by

$$\delta_{t,\tau}(p_t, \dots, p_{t+\tau-1}; w_1, \dots, w_{t+\tau-1}) := p_t(w_1, \dots, w_t) \cdot \ldots \cdot p_{t+\tau-1}(w_1, \dots, w_{t+\tau-1}).$$

In addition, we set $\delta_{t,0} := 1$. Thus $\delta_{t,\tau}$ indicates whether the agent will be in office in period $t+\tau$, provided that he is in office in period t. To simplify notation, we sometimes write functions that depend only on a finite number of past values as functions of the entire series of all past and future values, which means we will write expressions like $p_t(\mathbf{w})$ and $\delta_{t,\tau}(\mathbf{p}; \mathbf{w})$ instead of $p_t(w_1, \dots, w_t)$ and $\delta_{t,\tau}(p_t, \dots, p_{t+\tau-1}; w_1, \dots, w_{t+\tau-1})$, with $\mathbf{w} := (w_1, w_2, \dots)$.

Note that $\delta_{t,\tau}$ is defined for values of $\mathbf{p}$ and $\mathbf{w}$ that would, in fact, lead to the agent's recall before period t; in these cases, $\delta_{t,\tau}$ is to be understood as a hypothetical statement: Suppose the agent were in office in period t, would he persist until period $t+\tau$? We will use this hypothetical question to define the equilibrium concept.

The players' subjective probabilities $r_{t,\tau}$ and $s_{t,\tau}$ that the agent will be in office in period $t + \tau$ provided he is in office in period t are given by the expected value of $\delta_{t,\tau}$, conditioned on the players' assumptions about each other's behaviour and their information about the future. Hence, if we use P to denote the probability measure underlying the model, and if we assume that the players are correctly informed and rational when developing their expectations, we have, for $\tau \geq 1$,

$$r_{t,\tau} = s_{t,\tau} = \int \delta_{t,\tau}\big(p_t, \ldots, p_{t+\tau-1}; w_1, \ldots, w_{t-1}, e_t + A_t, \ldots, e_{t+\tau-1} + A_{t+\tau-1}\big)\,\mathrm{d}P.$$

Placing these probabilities in expressions (1) and (2), we obtain the players' expected future payoffs as functions $\mathfrak{v}_t$ and $\mathfrak{w}_t$ of the strategies $\mathbf{e}$ and $\mathbf{p}$ chosen by the players and of a vector of the principal's utility levels $(w_1, \ldots, w_{t-1})$, respectively.

We introduce some notation. The symbol $\Gamma(-\infty)$ is used to denote the reappointment game just defined. We use $\widetilde{\Sigma}_{\mathrm{P}}(-\infty)$ to denote the principal's strategy space, while $\widetilde{\Sigma}_{\mathrm{A}}$ stands for the agent's strategy space. The meaning of the argument $-\infty$ will become clear in Section 4, where we introduce threshold contracts. For the time being, it should be considered as a mere notational item.

Now we can introduce the equilibrium concept we shall be using:

Definition 1 (equilibrium)**.** *A pair* $(\mathbf{e}, \mathbf{p}) \in \widetilde{\Sigma}_{\mathrm{A}} \times \widetilde{\Sigma}_{\mathrm{P}}(-\infty)$ *is called an* equilibrium *of the reappointment game* $\Gamma(-\infty)$ *if for all* $t \in \mathbb{N}^*$ *and for all* $(w_1, \ldots, w_{t-1}) \in \mathbb{R}^{t-1}$ *the following conditions hold:*

(i) $\mathfrak{v}_t\big(\mathbf{e}, \mathbf{p}; w_1, \ldots, w_{t-1}\big) \geq \mathfrak{v}_t\big(\mathbf{e}', \mathbf{p}; w_1, \ldots, w_{t-1}\big)$ *for all* $\mathbf{e}' \in \widetilde{\Sigma}_{\mathrm{A}}$*,*

(ii) $\mathfrak{w}_t\big(\mathbf{e}, \mathbf{p}; w_1, \ldots, w_{t-1}\big) \geq \mathfrak{w}_t\big(\mathbf{e}, \mathbf{p}'; w_1, \ldots, w_{t-1}\big)$ *for all* $\mathbf{p}' \in \widetilde{\Sigma}_{\mathrm{P}}(-\infty)$*.*

The definition means that in equilibrium both players intend to behave optimally in any period—regardless of whether this point will actually be reached or not—and that their assumptions about the opponent's strategy conform to the strategy that is actually played. Thus, the equilibrium concept employed is, in fact, subgame-perfectness.

3. Stationary Markovian Strategies

3.1. Definition

The structure of the game remains the same as long as the agent is reappointed. Accordingly, we focus on equilibria where the players' behaviour is persistent. We call such equilibria stationary Markovian equilibria[3] and define them as follows:

Definition 2 (Markovian strategies, stationary strategies)**.**

(i) *A strategy* $\mathbf{e} = (e_1, e_2, \dots)$ *of the agent is called a* Markovian *strategy if the agent's behaviour does not depend on any previous utility level of the principal, i. e. if all the* e_t *are constant functions. A strategy* $\mathbf{p} = (p_1, p_2, \dots)$ *of the principal is called* Markovian *if no* p_t *depends on the principal's utility levels* $w_1, \dots, w_{t-1}$ *from earlier periods.*

(ii) *A Markovian strategy is called* stationary *if all elements of the strategy vector are equal. We use* Σ_{A} *to denote the set of all the agent's stationary Markovian strategies and* $\Sigma_{\mathrm{P}}(-\infty)$ *to denote the set of all the principal's stationary Markovian strategies.*

(iii) *An equilibrium* $(\mathbf{e}, \mathbf{p})$ *is called a Markovian equilibrium if both* $\mathbf{e}$ *and* $\mathbf{p}$ *are Markovian strategies. Analogously, an equilibrium* $(\mathbf{e}, \mathbf{p})$ *is called a stationary Markovian equilibrium if both* $\mathbf{e}$ *and* $\mathbf{p}$ *are stationary Markovian strategies.*

Note that the definition of stationarity is sensible in the Markovian case because the elements of the strategy vectors can be seen as choices of a constant value in $[\underline{e}; e^{\sup})$ (for the agent) or as functions of one argument (for the principal). Accordingly, "equality" of the elements across various periods is well-defined. The "Markovian strategy" notion is motivated by the fact that, with such strategies, a player's action is a response to the last action of the opponent only.

With Markovian strategies $\mathbf{e}$ and $\mathbf{p}$, the utility levels $\mathfrak{v}_t(\mathbf{e}, \mathbf{p}, \mathbf{w})$ and $\mathfrak{w}_t(\mathbf{e}, \mathbf{p}, \mathbf{w})$ do not depend on the noise levels realized previously, so in that case we can omit the third argument and simply write $\mathfrak{v}_t(\mathbf{e}, \mathbf{p})$ and $\mathfrak{w}_t(\mathbf{e}, \mathbf{p})$. Since *stationary* Markovian strategies are determined by a single function p or number e, we will simply write p or e for such strategies. If both the agent and the principal pursue stationary Markovian strategies, we simply write $\mathfrak{v}(e, p)$ and $\mathfrak{w}(e, p)$ instead of $\mathfrak{v}_t(e, p)$ and $\mathfrak{w}_t(e, p)$. Special stationary Markovian strategies are given by the constant functions

$$\mathsf{0}\colon \mathbb{R} \to \{0, 1\}, \quad w \mapsto 0,$$

[3] In Sections 1 and 3 of their paper, Haller and Lagunoff (2000) discuss the restrictiveness of Markovian strategies in great detail, and give a comprehensive literature overview.

under which the agent is never reappointed, regardless of the principal's utility level, and

$$\mathbb{1}\colon \mathbb{R} \to \{0,1\}, \quad w \mapsto 1,$$

under which the agent is always reappointed.

In the case of stationary Markovian strategies, many of the above formulae can be greatly simplified. In particular, we can frequently drop the index t. Using

$$\mathfrak{q}(e,p) := \int p(e+A)\,\mathrm{d}P \tag{3}$$

to denote the probability that the agent will be reappointed at the end of a period in which he was in office, we have $r_{t,\tau} = s_{t,\tau} = \mathfrak{q}(e,p)^{\tau}$.

We use $\mathsf{v}(e,q)$ to denote the agent's expected utility if he exerts effort e as long as he is in office and the probability of reappointment is q in each period. The expected utility amounts to

$$\begin{aligned}\mathsf{v}(e,q) &= (1-\beta)^{-1}v_* + \sum_{t=0}^{\infty} q^t\beta^t\big(v(e)-v_*\big)\\ &= (1-\beta)^{-1}v_* + \frac{v(e)-v_*}{1-q\beta}.\end{aligned} \tag{4}$$

The corresponding expected utility of the principal is denoted by $\mathsf{w}(e,q)$ and given by

$$\begin{aligned}\mathsf{w}(e,q) &= (1-\gamma)^{-1}w_* + \sum_{t=0}^{\infty} q^t\gamma^t\big(e-w_*\big)\\ &= (1-\gamma)^{-1}w_* + \frac{e-w_*}{1-q\gamma}.\end{aligned} \tag{5}$$

With these definitions, we have

$$\mathfrak{v}(e,p) = \mathsf{v}\big(e,\mathfrak{q}(e,p)\big) \quad \text{and} \quad \mathfrak{w}(e,p) = \mathsf{w}\big(e,\mathfrak{q}(e,p)\big). \tag{6}$$

3.2. Characteristics of equilibrium

As the following proposition states, stationary Markovian equilibrium can be characterized by considering only stationary Markovian deviations:

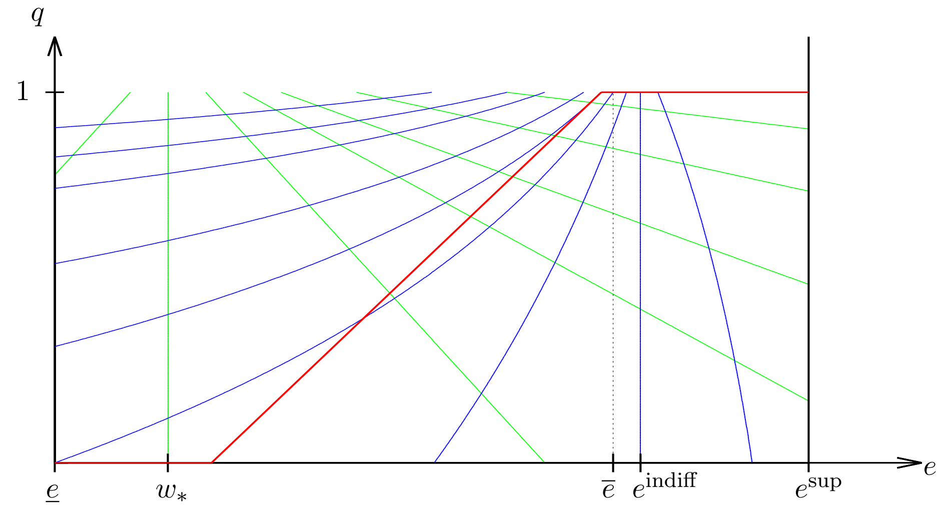

Figure 1: Indifference curves of the principal (green) and the agent (blue). The red line is the function $e \mapsto \mathfrak{q}(e, p)$ with a threshold strategy and for uniformly distributed noise (i. e. for A being uniformly distributed on some interval $[-\overline{a}; \overline{a}] \subseteq \mathbb{R}$).

Proposition 1. *A pair $(e^*, p^*) \in \Sigma_A \times \Sigma_P(-\infty)$ of stationary Markovian strategies is an equilibrium of the game $\Gamma(-\infty)$ if and only if $\mathfrak{v}(e^*, p^*) \geq \mathfrak{v}(e, p^*)$ for any stationary Markovian strategy $e \in \Sigma_A$ of the agent and $\mathfrak{w}(e^*, p^*) \geq \mathfrak{w}(e^*, p)$ for any stationary Markovian strategy $p \in \Sigma_P(-\infty)$ of the principal.*

The proposition reflects Bellman's principle of optimality[4]. If the opponent pursues a stationary Markovian strategy, the situation in which a player has to decide remains the same in each period. Hence, if the player could profitably deviate from his strategy in one period, it would be profitable to repeat this deviation in all subsequent periods. The proof of the proposition is based on this idea and is given in the Appendix A.1.

Together with Equations (4), (5), and (6), the proposition permits an instructive graphical illustration of the players' decision problems.

Consider Figure 1. The variables that appear in it will be explained shortly. The blue curves are indifference curves of the agent, along which v is constant. As can be seen from solving Equation (4) for q, they are copies of the graph of the function $e \mapsto v(e)$, mirrored and scaled vertically. If the agent's indifference curves were extended to values of q strictly larger than 1 by permitting such values of q in Equation (4) (which would

[4] "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" (Bellman, 1957, p. 83).

not make sense economically), they would intersect at the point $(e^{\text{indiff}}; 1/\beta)$, with e^{indiff} determined by the condition $v(e^{\text{indiff}}) = v_*$. The farther to the left an indifference curve is, the higher is its associated utility level $\mathsf{v}(\cdot,\cdot)$.

The indifference curves of the principal are described by Equation (5), with w being held constant. In Figure 1 they are represented by the green straight lines intersecting at the point $(w_*; 1/\gamma)$. The more a green line inclines to the right, the higher the represented utility level w.

By Equation (3), a strategy p of the principal leads to a function $\mathfrak{q}(\cdot, p)\colon e \mapsto \mathfrak{q}(e, p)$ that relates effort levels to reappointment probabilities. As a special case, we shall be examining threshold strategies, which we will now define. For any set C, we use $\mathbb{1}_C$ to denote its indicator function, i. e.

$$\mathbb{1}_C(x) := \begin{cases} 1 & \text{for } x \in C, \\ 0 & \text{for } x \notin C. \end{cases}$$

Definition 3. *A* threshold strategy *(with threshold $b \in [-\infty; +\infty]$) is a stationary Markovian strategy $p \in \Sigma_{\mathrm{P}}(-\infty)$ of the principal such that $p(w) = \mathbb{1}_{[b;\infty)}(w)$ for all $w \in \mathbb{R}$, i. e. the agent is reappointed if and only if the principal's period utility has reached threshold b.*

For thresholds $-\infty$ and $+\infty$, we obtain two special cases of a threshold strategy. If the threshold is $-\infty$, the principal always reappoints the agent—formally, we have $\mathbb{1}_{[-\infty;\infty)} = \mathsf{1}$. If the threshold is $+\infty$, the agent is never reappointed—we have $\mathbb{1}_{[+\infty;\infty)} = \mathsf{0}$. If p is a threshold strategy with a finite threshold $b \in (-\infty; +\infty)$, then the function $\mathfrak{q}(\cdot, p)$ is the cumulative distribution function of $-A$, shifted to the right by b units:

$$\mathfrak{q}(e, p) = \int \mathbb{1}_{e+A \geq b} \, \mathrm{d}P = P(-A \leq e - b) = F_{-A}(e - b).$$

The red line in Figure 1 shows the function $\mathfrak{q}(\cdot, p)$ with such a threshold strategy and for uniformly distributed noise.

A pair $(e, p) \in \Sigma_{\mathrm{A}} \times \Sigma_{\mathrm{P}}(-\infty)$ is a stationary Markovian equilibrium of the game $\Gamma(-\infty)$ if and only if (i) the point $\big(e, \mathfrak{q}(e, p)\big)$ is a contact point of the graph of $\mathfrak{q}(\cdot, p)$ (red line) and the uppermost indifference curve of the agent that is touched by the graph, *and* (ii) no reappointment strategy p' exists for which $\big(e, \mathfrak{q}(e, p')\big)$ is located on a higher indifference curve of the principal than $\big(e, \mathfrak{q}(e, p)\big)$.

Since by choosing the effort level $\underline{e}$ the agent can always ensure a utility level of at least $\mathsf{v}(\underline{e}, 0)$ for himself, the agent's utility in any equilibrium must be at least $\mathsf{v}(\underline{e}, 0)$. Graphically, this means that point $\big(e, \mathsf{q}(e,p)\big)$ must not lie below the indifference curve that runs through point $(\underline{e}, 0)$. In particular, there are no equilibria (e,p) with $e > \overline{e}$, where we use $\overline{e}$ to denote the effort level at which this indifference curve intersects the horizontal straight line defined by $q = 1$. Formally, $\overline{e}$ is determined by the equation

$$\mathsf{v}(\overline{e}, 1) = \mathsf{v}(\underline{e}, 0).$$

Since v is assumed to be strictly decreasing, since $v(\underline{e}) > v_*$, and since $\lim_{e \to e^{\sup}} v(e) = -\infty$, the effort level $\overline{e}$ is well-defined and unique; furthermore, $\underline{e} < \overline{e} < e^{\text{indiff}}$. Consider any $e \in [\underline{e}; e^{\sup})$. Then, we have $e \leq \overline{e}$ if and only if $\mathsf{v}(e,1) \geq \mathsf{v}(\underline{e}, 0)$, i. e. if and only if

$$v(\underline{e}) + \beta v_* \leq v(e) + \beta v(\underline{e}),$$

which is equivalent to

$$v(\underline{e}) - v(e) \leq \beta\big(v(\underline{e}) - v_*\big). \tag{7}$$

Hence, $e \leq \overline{e}$ if and only if the gain in utility from behaving sluggardly today, i. e. choosing $\underline{e}$ instead of e, is not higher than the gain from being in office in the next period and behaving sluggardly then.

Now we can give necessary and sufficient conditions for a stationary Markovian equilibrium. For this purpose we define

$$\mathcal{E} := [\underline{e}; \overline{e}] \setminus (\underline{e}; w_*).$$

Proposition 2. *Consider the game* $\Gamma(-\infty)$.

(i) Let (e,p) *be a stationary Markovian equilibrium. If* $e < w_*$*, then* $\mathsf{q}(e,p) = 0$*; if* $e > w_*$*, then* $\mathsf{q}(e,p) = 1$.

(ii) A "critical" effort level $e^{\text{crit}} \in \mathcal{E}$ *exists such that the following statements hold:*

- *(a) If* (e,p) *is a stationary Markovian equilibrium, then* $e \in \mathcal{E}$ *with* $e \leq e^{\text{crit}}$.
- *(b) For every* $e \in \mathcal{E}$ *with* $e \leq e^{\text{crit}}$*, a strategy* p *exists such that the pair* (e,p) *is a stationary Markovian equilibrium.*

(iii) If $w_* \geq \underline{e}$*, then the pair* $(\underline{e}, \mathsf{0})$ *is a stationary Markovian equilibrium.*

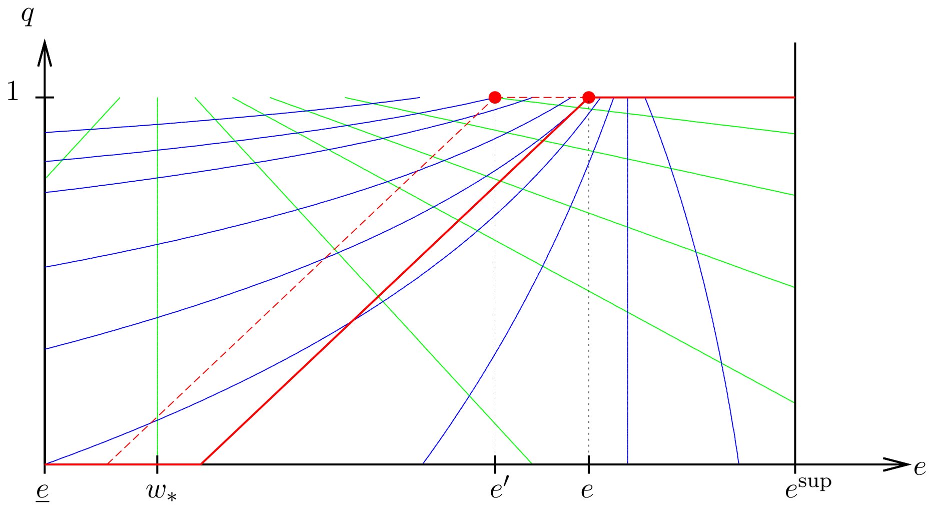

Figure 2: If the red line is shifted to the left, the position of the optimum relative to the red line does not change.

(iv) If $w_* \leq \underline{e}$, then the pair $(\underline{e}, 1)$ is a stationary Markovian equilibrium.

Note that in Part (ii) we did *not* state $e^{\text{crit}} > w_*$; hence the set of effort levels $e \in \mathcal{E}$ with $e \leq e^{\text{crit}}$ may degenerate to $\{\underline{e}\}$ or $\{\underline{e}, w_*\}$.

Proof. Part (i): Consider a pair (e,p) with $e < w_*$. Since $\mathfrak{w}(e,0) > \mathfrak{w}(e,p)$ for any p with $\mathfrak{q}(e,p) > 0$ (which reflects the fact that the principal strictly prefers his "outside option" of w_* to the agent's exerting effort below w_*), a necessary condition for (e,p) to be an equilibrium is $\mathfrak{q}(e,p) = 0$.

If $e > w_*$, the principal is strictly better off with the agent than without him, so $\mathfrak{w}(e,1) > \mathfrak{w}(e,p)$ for any p with $\mathfrak{q}(e,p) < 1$, which means a necessary condition for (e,p) to be an equilibrium is $\mathfrak{q}(e,p) = 1$.

Part (ii): Let (e,p) be an equilibrium with $e < w_*$. From Part (i) we know that $\mathfrak{q}(e,p) = 0$. This implies $\mathfrak{v}(\underline{e},0) > \mathfrak{v}(e,p)$ for $e > \underline{e}$. Thus, for a pair (e,p) with $e < w_*$ to be an equilibrium, e must equal $\underline{e}$. In addition, for $e > \overline{e}$ and any p, we have $\mathfrak{v}(e,p) < \mathfrak{v}(\overline{e},1) = \mathfrak{v}(\underline{e},0) \leq \mathfrak{v}(\underline{e},p)$; hence (e,p) will not be an equilibrium. These considerations show that $e \in \mathcal{E}$ is a necessary condition for a pair (e,p) to be an equilibrium.

We will demonstrate the following:

Fact. *Let (e, p) be an equilibrium with $w_* < e \le \overline{e}$ and let $e' \in [w_*; e]$. Then, a strategy p' of the principal exists such that (e', p') is an equilibrium as well.*

From this fact one can readily conclude that the set of effort levels within $[w_*; \overline{e}]$ which can occur in equilibrium is either empty or a non-empty interval of the form $[w_*; e^{\text{crit}})$ or $[w_*; e^{\text{crit}}]$, for some e^{crit}. Lemma 1 below, which we shall prove without recourse to Proposition 2, tells us that such an interval must be closed from above; hence the Fact and Lemma 1 together yield Part (ii).

We prove the Fact, using Figure 2 to develop the argument. Since the effort level e is optimal for the agent, the point in which the red line touches the highest of the agent's indifference curves is $\big(e, \mathfrak{q}(e, p)\big)$. From Part (i) we know that necessarily $\mathfrak{q}(e, p) = 1$, which means that the point is associated with a reappointment probability of one. When the solid red line is shifted to the left by $e - e'$ units (dotted red line), the agent's optimal point is shifted to the left by $e - e'$ units as well—to the point $(e', 1)$—, since the agent's indifference curves become only flatter as one moves to the left. Thus, if $\big(e, \mathfrak{q}(e, p)\big) = (e, 1)$ is optimal for the agent under the reappointment scheme represented by the solid red line, $(e', 1)$ is optimal for the agent under the reappointment scheme represented by the dotted red line. A shift of the red line corresponds to a shift of p. Hence we should shift p by $e' - e$ units to attain e' as an equilibrium effort level.

Formally, we define p' by $p'(w) := p(w - e' + e)$ and show that (e', p') is an equilibrium. We first prove that $\mathfrak{v}(\tilde{e}', p') \le \mathfrak{v}(e', p')$ for any strategy $\tilde{e}'$ of the agent. For $\tilde{e}' \ge e'$ the inequality is obvious; hence we are restricted to the case $\tilde{e}' < e'$. Let $\tilde{e} := \tilde{e}' + e - e'$. Since (e, p) is an equilibrium, $\mathfrak{v}(\tilde{e}, p) \le \mathfrak{v}(e, p)$, which implies

$$\frac{1 - \beta \mathfrak{q}(e, p)}{1 - \beta \mathfrak{q}(\tilde{e}, p)} \le \frac{v(e) - v_*}{v(\tilde{e}) - v_*}\,.$$

By the strict concavity of v, the right-hand side is strictly less than $\big(v(e') - v_*\big)/\big(v(\tilde{e}') - v_*\big)$. Hence, since $\mathfrak{q}(e, p) = \mathfrak{q}(e', p')$ and $\mathfrak{q}(\tilde{e}, p) = \mathfrak{q}(\tilde{e}', p')$,

$$\frac{1 - \beta \mathfrak{q}(e', p')}{1 - \beta \mathfrak{q}(\tilde{e}', p')} < \frac{v(e') - v_*}{v(\tilde{e}') - v_*}\,,$$

and, therefore, $\mathfrak{v}(\tilde{e}', p') < \mathfrak{v}(e', p')$.[5]

[5] For this reasoning, a reappointment probability of one is not necessary. It is generally true that if the red line is shifted to the left, the best response only moves to the right (or does not change), *relative* to the red line. Thus a best response will never drop by more than δ units due to a left-shift of p by δ units. This argument underlies Part (iv) of Lemma 2.

Since $\mathfrak{q}(e', p') = 1$ and $e' \geq w_*$, we have $\mathfrak{w}(e', \tilde{p}') \leq \mathfrak{w}(e', p')$ for any strategy $\tilde{p}'$ of the principal. Thus, the principal will not want to deviate from (e', p').

Part (iii): Since $\underline{e}$ is the smallest effort level the agent can choose, we have $v(e') \leq v(\underline{e})$ for all e'. This implies $\mathfrak{v}(e', 0) \leq \mathfrak{v}(\underline{e}, 0)$; hence the agent, who assumes that the principal will never reappoint him, has no incentive to deviate from choosing $\underline{e}$. If $w_* \geq \underline{e}$, then $\mathfrak{w}(\underline{e}, 0) \geq \mathfrak{w}(\underline{e}, p)$ for all p; hence the principal will not deviate, either. This shows that $(\underline{e}, 0)$ is an equilibrium.

Part (iv): Similar to the last paragraph, we have $\mathfrak{v}(e', 1) \leq \mathfrak{v}(\underline{e}, 1)$ for all e'. Further, if $w_* \leq \underline{e}$, we have $\mathfrak{w}(\underline{e}, 1) \geq \mathfrak{w}(\underline{e}, p)$ for all p. Hence, $(\underline{e}, 1)$ is an equilibrium. □

It remains to state Lemma 1, used in the proof of Proposition 2:

Lemma 1. *Consider a sequence of stationary Markovian equilibria $\big((e_n, p_n)\big)_{n=1}^{\infty}$ such that $e_n \in (\underline{e}; \overline{e}]$ for all n. Suppose the sequence $(e_n)_{n=1}^{\infty}$ of effort levels is strictly increasing and converges to some effort level $\tilde{e} \in (\underline{e}; \overline{e}]$. Then a strategy $\tilde{p} \in \Sigma_{\mathrm{P}}(-\infty)$ exists such that $(\tilde{e}, \tilde{p})$ is an equilibrium.*

The Lemma is proved in Appendix A.1.

3.3. Example: Uniform distribution

To illustrate the results obtained so far, we derive the stationary Markovian equilibria in the case of uniform noise distribution. We will pick this example up again in Section 5.2, when we illustrate the effects of a threshold contract. Note that as the function v is assumed to be concave, the left-hand derivative $\partial^- v$ exists everywhere in $(\underline{e}; e^{\sup})$.

Proposition 3. *Let the noise A be uniformly distributed on an interval $[-\overline{a}; \overline{a}] \subseteq \mathbb{R}$ for some $\overline{a} > 0$. Let $e^* \in (\underline{e}; e^{\mathrm{indiff}})$. A stationary Markovian strategy p^* such that (e^*, p^*) is an equilibrium of the game $\Gamma(-\infty)$ exists if and only if $w_* \leq e^* \leq \overline{e}$ and*

$$-\frac{\partial^- v(e^*)}{v(e^*) - v_*} \leq \frac{\beta}{1-\beta} \cdot \frac{1}{2\overline{a}}\,. \tag{8}$$

p^* can be chosen as $p^*(w) = \mathbb{1}_{[e^* - \overline{a}; \infty)}(w)$.

Proof. Sufficiency. Consider an arbitrary effort level $e^* \in [w_*; \overline{e}]$ satisfying Condition (8). We will demonstrate that (e^*, p^*) with

$$p^* = \mathbb{1}_{[e^* - \overline{a}; \infty)} \tag{9}$$

is an equilibrium. Since $\mathfrak{q}(e^*, p^*) = 1$ and, by assumption, $e^* \geq w_*$, the strategy p^* is a best response of the principal to the agent's playing e^*.

It remains to be shown that e^* is a best response of the agent to the strategy p^* of the principal. We shall do this with the help of Figure 1. The function $e \mapsto \mathfrak{q}(e, p^*)$ is depicted by the red line. Since p^* is given by (9), the kinks of the red curve are located at $e^* - 2\overline{a}$ and e^*. Recall that the agent's best response to p^* is determined by that point on the red curve which yields the highest utility for the agent.

The slope of the agent's indifference curve running through the point $(e^*, 1)$ is given by

$$-\frac{1-\beta}{\beta} \cdot \frac{\partial^- v(e^*)}{v(e^*) - v_*} \,.$$

Thus, Inequality (8) ensures that the graph of $e \mapsto \mathfrak{q}(e, p^*)$ (the red line) does not cross the agent's indifference curve there. Since the agent's indifference curves are strictly convex and since the red line is concave on the interval $[e^* - 2\overline{a}; \infty)$ (i. e. to the right of the first kink), Condition (8) guarantees that e^* is the unique best response of the agent on the interval $[e^* - 2\overline{a}; \infty)$.

The additional condition $e^* \leq \overline{e}$ ensures that $\mathfrak{v}(e^*, p^*) \geq \mathfrak{v}(\underline{e}, p^*)$, which implies that no effort level left to the kink is better for the agent than e^*.

Necessity. Let (e^*, p^*) be a stationary Markovian equilibrium with $e^* \in (\underline{e}; e^{\sup})$. From Proposition 2 it follows that $w_* \leq e^* \leq \overline{e}$. If Inequality (8) is violated, the blue indifference curve passing through $(e^*, 1)$ is steeper than the red line, which means that the agent can improve his utility by deviating from e^* towards lower effort levels. Hence, Condition (8) is necessary. □

The left-hand side of Inequality (8) is monotonically increasing in e^*. The critical effort level e^{crit} from Proposition 2 is, therefore, the supremum of the values within $\mathcal{E}$ that satisfy Inequality (8). We conclude the example by observing that e^{crit} will decline if the variance of the noise increases, i. e. if $\bar{a}$ is larger. Hence for $\overline{a}$ suitably high, equilibrium-supporting effort levels above $\underline{e}$ do not exist.

4. Threshold Contracts

We now turn to the examination of threshold contracts. In Section 4.1 we explain their consequences for the reappointment game. We focus on equilibria in which the principal pursues a threshold strategy. Such equilibria are called *threshold equilibria*.[6] In Section 4.2 we start analysing threshold contracts. In Section 4.3 we introduce the notion of a welfare function and show the existence of a welfare-maximizing threshold equilibrium. Section 4.4 addresses dominance and uniqueness questions. In Section 4.5 we examine the maximization of the principal's utility, and Section 4.6 deals with utility maximization from the agent's point of view.

4.1. The reappointment game with a threshold contract

We assume that the agent can commit to a reappointment threshold at the beginning of the game. Such a commitment is called a *threshold contract*. It works as follows: The agent announces a certain threshold $\tau \in [-\infty; +\infty]$. The principal is not allowed to reappoint the agent after a period in which the principal's utility falls below τ, even if he would like to do so. The commitment, however, is not binding in the other direction, i. e. there is no obligation for the principal to reappoint the agent if the threshold has been reached. We use $\Gamma(\tau)$ to denote the reappointment game with threshold τ. Since committing to $\tau = -\infty$ is equivalent to no commitment at all, this notation is consistent with our previous definition of $\Gamma(-\infty)$ denoting the reappointment game without a threshold contract.

What are the consequences of a threshold contract? Formally, the reappointment threshold τ restricts the principal's strategy space to strategies $(p_1, p_2, \dots) \in \widetilde{\Sigma}_{\mathrm{P}}(-\infty)$ that for all $t \geq 1$ satisfy the constraint

$$p_t(w_1, \dots, w_t) = 0 \quad \text{for } w_t < \tau. \tag{10}$$

We use $\widetilde{\Sigma}_{\mathrm{P}}(\tau)$ to denote the set of all such strategies; thus $\widetilde{\Sigma}_{\mathrm{P}}(\tau)$ is the principal's strategy space in the game $\Gamma(\tau)$. The agent's strategy space is not affected by the threshold contract and is given by $\widetilde{\Sigma}_{\mathrm{A}}$.

[6] In a technical companion paper (Becker and Gersbach, 2011), we provide sufficient conditions on the distribution of the noise which ensure that the restriction to threshold strategies does not limit the scope of the principal.

For stationary Markovian strategies p, Constraint (10) becomes

$$p(w) = 0 \quad \text{for } w < \tau,$$

which is equivalent to

$$p(w) \leq \mathbb{1}_{[\tau;\infty)}(w) \quad \text{for all } w. \tag{11}$$

We use $\Sigma_{\mathrm{P}}(\tau)$ to denote the set of all stationary Markovian strategies of the principal that satisfy this constraint.

By replacing the argument $-\infty$ by τ everywhere in Definition 1, we can generalize the definition of equilibrium given there to $\Gamma(\tau)$ for arbitrary $\tau \in [-\infty; +\infty]$. Thus it is clear what we mean by an *equilibrium of the game* $\Gamma(\tau)$. Similarly, Proposition 1 can be generalized to all games $\Gamma(\tau)$. The proof carries over readily (with one single change described in footnote 9 on page 51).

A threshold contract can have two effects:

1. Equilibria can disappear as they involve a reappointment strategy of the principal that is not possible under the commitment.
2. New equilibria can arise as the commitment excludes reappointment strategies that would be profitable deviations for the principal.

As the following proposition clarifies, the first effect does indeed describe the only reason why equilibria may disappear.

Proposition 4. *Consider thresholds σ, τ with $-\infty \leq \sigma \leq \tau \leq \infty$. If $(e, p) \in \Sigma_{\mathrm{A}} \times \Sigma_{\mathrm{P}}(\tau)$ is an equilibrium of the game $\Gamma(\sigma)$, then (e, p) is an equilibrium of the game $\Gamma(\tau)$.* □

Since the statement is obvious, a detailed proof is not needed. From the proposition, together with Part (iii) of Proposition 2, we immediately obtain a corollary:

Corollary 1. *Suppose $w_* \geq \underline{e}$. Then for all $\tau \in [-\infty; +\infty]$, the pair $(\underline{e}, 0)$ is an equilibrium of the game $\Gamma(\tau)$.* □

The equilibrium $(\underline{e}, 0)$ continues to exist in the presence of a threshold contract, as a threshold contract never prevents the principal from choosing $p = 0$.

In Proposition 2 we saw that, without a threshold contract, any stationary Markovian equilibrium (e^*, p^*) with an effort level strictly above w_* necessarily involves a reappointment probability $\mathfrak{q}(e^*, p^*)$ of one. The reason for this is intuitive. The principal is willing to reappoint the agent with a probability of one if he expects the agent to exert effort higher than w_*, since his expected utility from keeping the agent is higher than the expected utility from firing him. If there is no threshold contract, the principal can achieve a reappointment probability of one by switching to the strategy of always reappointing (i. e. formally the strategy $\mathbb{1}$). This means that from any strategy profile (e, p) with $e > w_*$ and $\mathfrak{q}(e, p) < 1$ the principal can deviate profitably to the strategy $\mathbb{1}$; hence (e, p) cannot be an equilibrium.

Under a threshold contract, equilibria involving a reappointment probability of strictly less than one may become possible even if the effort level is strictly greater than w_*, since the threshold may prevent the principal from switching to a higher probability of reappointment in cases in which he would like to do so. In equilibrium, nevertheless, the principal will reappoint the agent with the highest probability permitted by the threshold contract. These considerations lead to the following proposition:

Proposition 5. *Consider the game $\Gamma(\tau)$ for some $\tau \in [-\infty; +\infty]$. Let (e, p) be a stationary Markovian equilibrium. If $e < w_*$, then $e = \underline{e}$ and $\mathfrak{q}(e, p) = 0$; if $e > w_*$, then $\mathfrak{q}(e, p) = \mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)})$.* □

Both Corollary 1 and Proposition 5 generalize some of the statements of Proposition 2. Proposition 5 is a generalization of Part (i) of Proposition 2, and Corollary 1 is a generalization of Part (iii). Parts (ii) and (iv) of Proposition 2 do not carry over to the case of non-trivial threshold contracts.[7]

4.2. Threshold strategies, threshold equilibria, and best responses

Our focus is on equilibria in which the principal pursues a threshold strategy; we are going to call such an equilibrium a *threshold equilibrium*:

Definition 4. *A stationary Markovian equilibrium $(e, p) \in \Sigma_{\mathrm{A}} \times \Sigma_{\mathrm{P}}(\tau)$ of the game $\Gamma(\tau)$ with $\tau \in [-\infty; +\infty]$ is called a* threshold equilibrium *of $\Gamma(\tau)$ if p is a threshold strategy (i. e. if $p = \mathbb{1}_{[b;\infty)}$ for some $b \in [\tau; +\infty]$).*

[7] See Section II.7.4 of Becker (2011) for counter-examples.

For any threshold b, we use $\mathcal{R}(b)$ to denote the set of the agent's best responses to the threshold strategy $\mathbb{1}_{[b;\infty)}$, i. e.

$$\mathcal{R}(b) := \left\{ e \in [\underline{e}; e^{\sup}) \;\middle|\; \mathfrak{v}(e, \mathbb{1}_{[b;\infty)}) \geq \mathfrak{v}(e', \mathbb{1}_{[b;\infty)}) \text{ for all } e' \in [\underline{e}; e^{\sup}) \right\}.$$

Obviously, $\mathcal{R}(b) \subseteq [\underline{e}; \overline{e}]$ for all $b \in [-\infty; +\infty]$. We are now in a position to characterize the threshold equilibria of the game $\Gamma(\tau)$:

Proposition 6. *For any $\tau \in [-\infty; +\infty]$, consider the game $\Gamma(\tau)$. A pair $(e, \mathbb{1}_{[b;\infty)})$ with $e \in [\underline{e}; e^{\text{indiff}})$ and $b \in [\tau; \infty]$ is an equilibrium of $\Gamma(\tau)$ if and only if $e \in \mathcal{R}(b)$ and one of the following conditions holds:*

(i) $e > w_*$ *and* $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = \mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)})$,

(ii) $e = w_*$,

(iii) $e = \underline{e} < w_*$ *and* $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = 0$.

In particular, any pair $(e, \mathbb{1}_{[\tau;\infty)})$ with $e \in \mathcal{R}(\tau)$, $e \geq w_*$, is an equilibrium of $\Gamma(\tau)$.

Proof. The "only if" part is clear by the definition of $\mathcal{R}(\cdot)$ and by Proposition 5. We now turn to the "if" part. The condition $e \in \mathcal{R}(b)$ means that the agent does not have an incentive to deviate from e. Now suppose the principal had the incentive to deviate to some stationary Markovian strategy p, which would mean that $\mathfrak{w}(e, p) > \mathfrak{w}(e, \mathbb{1}_{[b;\infty)})$. For $e > w_*$, this would imply $\mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)}) \geq \mathfrak{q}(e, p) > \mathfrak{q}(e, \mathbb{1}_{[b;\infty)})$, violating the assumption $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = \mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)})$. For $e = w_*$, we trivially have $\mathfrak{w}(e, p) = w_* = \mathfrak{w}(e, \mathbb{1}_{[b;\infty)})$. For $e = \underline{e} < w_*$, the inequality $\mathfrak{w}(e, p) > \mathfrak{w}(e, \mathbb{1}_{[b;\infty)})$ implies $\mathfrak{q}(\underline{e}, p) < \mathfrak{q}(\underline{e}, \mathbb{1}_{[b;\infty)})$, which is impossible if $\mathfrak{q}(\underline{e}, \mathbb{1}_{[b;\infty)}) = 0$. □

The proposition enables us to answer the question which strategy profiles $(e, \mathbb{1}_{[b;\infty)})$ are a threshold equilibrium of any of the games $\Gamma(\tau)$: Let $\mathcal{T}$ be the set of all such strategy profiles, i. e. let

$$\mathcal{T} := \Big\{ (e, \mathbb{1}_{[b;\infty)}) \in \Sigma_{\text{A}} \times \Sigma_{\text{P}}(-\infty) \;\Big|\; \exists\, \tau \in [-\infty; +\infty] : (e, \mathbb{1}_{[b;\infty)}) \text{ is an equilibrium of } \Gamma(\tau) \Big\}.$$

The above proposition yields the following corollary:

Corollary 2. *For* $\underline{e} \geq w_*$*, we have*

$$\mathcal{T} = \bigcup_{\tau \in [-\infty;+\infty]} \big\{ (e, \mathbb{1}_{[\tau;\infty)}) \bigm| e \in \mathcal{R}(\tau) \big\},$$

and for $\underline{e} < w_*$*, we have*

$$\mathcal{T} = \bigcup_{\tau \in [-\infty;+\infty]} \Big\{ (e, \mathbb{1}_{[\tau;\infty)}) \Bigm| e \in \mathcal{R}(\tau),\ e \geq w_* \Big\}$$
$$\cup \bigcup_{\tau \in [-\infty;+\infty]} \Big\{ (\underline{e}, \mathbb{1}_{[\tau;\infty)}) \Bigm| \underline{e} \in \mathcal{R}(\tau),\ \mathfrak{q}(\underline{e}, \mathbb{1}_{[\tau;\infty)}) = 0 \Big\}.$$

Proof. Proposition 6 tells us that the sets on the right-hand sides of the equations are contained in $\mathcal{T}$. We demonstrate the reverse inclusion, i. e. we show that $\mathcal{T}$ is contained in the sets on the right-hand sides of the equations. Consider any element (e, p) of $\mathcal{T}$. Then, thresholds $b, \tau \in [-\infty; +\infty]$ exist such that $p = \mathbb{1}_{[b;\infty)}$ and (e, p) is an equilibrium of $\Gamma(\tau)$. By Proposition 4, (e, p) is an equilibrium of $\Gamma(b)$ as well. Proposition 6 yields $e \in \mathcal{R}(b)$ and, further, (A) $e \geq w_*$, or (B) $e = \underline{e} < w_*$ and $\mathfrak{q}(\underline{e}, \mathbb{1}_{[b;\infty)}) = 0$. □

It is useful to consider the "joint graph" of $\mathcal{R}(\cdot)$ and $\mathfrak{q}(\cdot, \mathbb{1}_{[\cdot;\infty)})$, by which we mean the set

$$\mathcal{G} := \Big\{ (b, e, q) \in [-\infty; +\infty] \times [\underline{e}; \overline{e}] \times [0; 1] \Bigm| e \in \mathcal{R}(b),\ q = \mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) \Big\}.$$

The sets $\mathcal{G}$ and $\mathcal{T}$ are closely related: With the definition

$$\mathcal{G}_{\mathcal{T}} := \Big\{ (b, e, q) \in \mathcal{G} \Bigm| e \geq w_* \text{ or } \big(e = \underline{e} \text{ and } q = 0\big) \Big\},$$

Corollary 2 yields

$$\mathcal{T} = \big\{ (e, \mathbb{1}_{[b;\infty)}) \bigm| \exists q \in [0; 1] : (b, e, q) \in \mathcal{G}_{\mathcal{T}} \big\}. \tag{12}$$

In the next lemma, we summarize several properties of $\mathcal{R}(\cdot)$, $\mathcal{G}$, and $\mathcal{G}_{\mathcal{T}}$:

Lemma 2 (Properties of $\mathcal{R}(\cdot)$, $\mathcal{G}$, and $\mathcal{G}_{\mathcal{T}}$)**.**

(i) For each $b \in [-\infty; +\infty]$*, the set* $\mathcal{R}(b)$ *is non-empty and compact.*

(ii) $\mathcal{R}(-\infty) = \mathcal{R}(+\infty) = \{\underline{e}\}$ *and* $\lim_{b \to -\infty} \big(\max \mathcal{R}(b)\big) = \lim_{b \to +\infty} \big(\max \mathcal{R}(b)\big) = \underline{e}$*.*

(iii) The sets $\mathcal{G}$ *and* $\mathcal{G}_{\mathcal{T}}$ *are compact.*

(iv) *Consider* $b', b'' \in (-\infty; \infty)$ *with* $b' < b''$*. Then,* $\min \mathcal{R}(b') \geq \max \mathcal{R}(b'') - (b'' - b')$*. In particular,* $\mathfrak{q}(e', \mathbb{1}_{[b';\infty)}) \geq \mathfrak{q}(e'', \mathbb{1}_{[b'';\infty)})$ *for each* $e' \in \mathcal{R}(b')$ *and each* $e'' \in \mathcal{R}(b'')$*.*

(v) *Consider any threshold* b *and any* $e \in \mathcal{R}(b)$*. Then, for all* e' *with* $\underline{e} \leq e' \leq e$*, some* $b' \leq b$ *exists such that* $e' \in \mathcal{R}(b')$*.*

The proof is given in Appendix A.2. Note that the set $[-\infty; +\infty]$ is the *extended* real line; hence Part (iii) of the lemma does *not* imply that $\mathcal{G}$ is bounded—which would be obviously wrong, since the set $\mathcal{G}$ contains the points $(-\infty, \underline{e}, 1)$ and $(+\infty, \underline{e}, 0)$. Part (i) of the lemma enables us to prove the existence of equilibrium for the game $\Gamma(\tau)$:

Corollary 3. *For each* $\tau \in [-\infty; +\infty]$*, the game* $\Gamma(\tau)$ *possesses a threshold equilibrium.*

Proof. If $w_* \geq \underline{e}$, Corollary 1 tells us that for each $\tau \in [-\infty; +\infty]$, the pair $(\underline{e}, \mathsf{0})$ is an equilibrium of $\Gamma(\tau)$. Consider now the case $w_* < \underline{e}$ and any $\tau \in [-\infty; +\infty]$. By Part (i) of Lemma 2, a best response $e \in \mathcal{R}(\tau)$ exists. By Proposition 6, the pair $(e, \mathbb{1}_{[\tau;\infty)})$ is an equilibrium. □

Part (iv) of Lemma 2 yields a statement on the "generic" uniqueness of the agent's best response:[8]

Corollary 4. *There are at most countably many* $b \in [-\infty; +\infty]$ *for which* $\#\mathcal{R}(b) > 1$*.*

Proof. We prove the Corollary by contradiction, using the notation

$$d(b) := \max \mathcal{R}(b) - \min \mathcal{R}(b).$$

Suppose the statement is wrong. Then we can find $\underline{b}, \overline{b} \in (-\infty; +\infty)$ such that the set

$$\left\{ b \in (\underline{b}; \overline{b}) \;\middle|\; d(b) > 0 \right\}$$

is uncountable. Hence, for each $c \in \mathbb{R}$, we can find finitely many points $b_1, \ldots, b_N$ with $\underline{b} < b_1 < b_2 < \cdots < b_N < \overline{b}$ such that

$$\sum_{n=1}^{N} d(b_n) > c.$$

[8]Note that Corollary 4 is in the spirit of similar "genericity" statements as, for example, derived by Haller and Lagunoff (2000).

In particular, we can find such points for

$$c := \bar{b} - \underline{b} - \max \mathcal{R}(\bar{b}) + \min \mathcal{R}(\underline{b}).$$

Part (iv) of Proposition 2 yields

$$\begin{aligned}
\bar{b} - \underline{b} &= \big(\bar{b} - b_N\big) + \big(b_N - b_{N-1}\big) + \dots + \big(b_2 - b_1\big) + \big(b_1 - \underline{b}\big) \\
&\geq \big(\max \mathcal{R}(\bar{b}) - \min \mathcal{R}(b_N)\big) + \big(\max \mathcal{R}(b_N) - \min \mathcal{R}(b_{N-1})\big) + \dots \\
&\qquad + \big(\max \mathcal{R}(b_2) - \min \mathcal{R}(b_1)\big) + \big(\max \mathcal{R}(b_1) - \min \mathcal{R}(\underline{b})\big) \\
&= \max \mathcal{R}(\bar{b}) - \min \mathcal{R}(\underline{b}) + \sum_{n=1}^{N} d(b_n) \\
&> \max \mathcal{R}(\bar{b}) - \min \mathcal{R}(\underline{b}) + c \\
&= \bar{b} - \underline{b},
\end{aligned}$$

which is the desired contradiction. □

4.3. Welfare

We introduce the notion of a welfare function.

Definition 5. *A* welfare function *is a continuous function* $H\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *that is weakly increasing, that is for which*

$$H(x, y) \geq H(x', y') \textit{ if } x \geq x' \textit{ and } y \geq y'.$$

A welfare function H is used to aggregate the expected utilities of the agent (first argument) and the principal (second argument). If a pair (e, p) of stationary Markovian strategies is played, public welfare shall be given by

$$\mathfrak{H}_H(e, p) := H\big(\mathfrak{v}(e, p), \mathfrak{w}(e, p)\big).$$

When it is clear to which welfare function H we refer, we drop the index H of $\mathfrak{H}$.

Note that we do not require a welfare function to be strictly increasing. In particular, our analysis applies to the function $H(x, y) = y$, in which welfare is determined solely by the principal's utility (i. e. $\mathfrak{H} \equiv \mathfrak{w}$). We shall put special attention to this case in Section 4.5. Contrariwise, maximizing the welfare function $H(x, y) = x$ puts the focus on the utility of the agent. As we shall, however, observe later, this case leads to fairly uninteresting results, due to the asymmetric structure of threshold contracts, which enforce deselection, but cannot ensure reappointment.

A compactness argument immediately yields the existence of a welfare-maximizing threshold equilibrium:

Corollary 5 (Existence of a welfare-maximizing threshold equilibrium)**.** *The set*

$$\left\{\mathfrak{H}(e,p) \;\middle|\; (e,p)\in\mathcal{T}\right\}$$

is compact and non-empty. In particular, it contains a maximum.

Proof. The set $\left\{\mathfrak{H}(e,p) \mid (e,p)\in\mathcal{T}\right\}$ is not empty because, by Corollary 3, the set $\mathcal{T}$ is not empty. By Equation (12), we have

$$\left\{\mathfrak{H}(e,p) \;\middle|\; (e,p)\in\mathcal{T}\right\} = \left\{H\big(\mathsf{v}(e,q),\mathsf{w}(e,q)\big) \;\middle|\; (b,e,q)\in\mathcal{G}_{\mathcal{T}}\right\}. \tag{13}$$

Since the function $[-\infty;+\infty]\times[\underline{e};\overline{e}]\times[0;1]\to\mathbb{R}$, $(b,e,q)\mapsto H\big(\mathsf{v}(e,q),\mathsf{w}(e,q)\big)$, is continuous and, by Part (iii) of Lemma 2, the set $\mathcal{G}_{\mathcal{T}}$ is compact, the assertion follows. □

The corollary tells us that we can find a threshold contract involving some threshold $\hat{\tau}\in[-\infty;+\infty]$ and some threshold equilibrium $(\hat{e},\hat{p})$ of the game $\Gamma(\hat{\tau})$ such that public welfare in the equilibrium $(\hat{e},\hat{p})$ is at least as high as it would be in any threshold equilibrium under any threshold contract.

This result, however, is only moderately useful. In particular, we face the following issues:

1. The said equilibrium may involve effort level $\underline{e}$ or w_* and a low reappointment probability.
2. It is possible that $\hat{\tau}=-\infty$, i. e., there is no threshold contract leading to an improvement over what can be achieved without a threshold contract.
3. The game $\Gamma(\hat{\tau})$ may possess multiple threshold equilibria, some of which may be welfare-inferior to $(\hat{e},\hat{p})$.

The first of these problems is due to the fact that a threshold contract does not enforce reappointment of the agent and thus cannot be a measure for ruling out bad equilibria (equilibria of reciprocal distrust) in which the agent drags his feet because he does not expect to be reappointed and the principal hesitates to reappoint the agent because keeping the agent does not promise a strict benefit over deselection. Indeed, Corollary 1 shows that the bad equilibrium at $\underline{e}$ cannot be avoided. Further, it is not hard to construct an example in which each threshold contract that yields an equilibrium beyond $\underline{e}$ also induces an equilibrium at w_*. We will discuss the first and the second of the above points in Section 6. In the next section, we focus on the third point.

4.4. Welfare-maximizing threshold contracts

As we have seen, Corollary 5 ensures the existence of a welfare-maximizing threshold equilibrium $(\hat{e}, \hat{p})$, which is an equilibrium of some game $\Gamma(\hat{\tau})$. Unfortunately, $\Gamma(\hat{\tau})$ may well possess other threshold equilibria, which may be strictly welfare-inferior. If $w_* \geq \underline{e}$, it will possess the equilibrium $(\underline{e}, 0)$, which we cannot get rid of. As this equilibrium is Pareto-dominated by $(\hat{e}, \hat{p})$, one could reasonably argue that the players will resort to the dominating equilibrium.

Dominance criteria, however, still do not guarantee uniqueness. In Section 5.2 we shall re-consider the example from Section 3.3 (uniform noise distribution) and show that each game $\Gamma(\tau)$ in which $(\hat{e}, \hat{p})$ is an equilibrium may have a multiplicity of threshold equilibria that cannot be ordered in the Pareto sense.

This phenomenon may occur because in the game $\Gamma(\hat{\tau})$, all threshold strategies $\mathbb{1}_{[b;\infty)}$ with $b \geq \hat{\tau}$ are admissible strategies and thus the principal is not required to pursue the threshold strategy $\mathbb{1}_{[\hat{\tau};\infty)}$; those other strategies, however, may lead to equilibria which are utility-superior for the principal, but utility-inferior for the agent.

This cannot happen under two additional conditions. First, the set-up must be such that threshold contracts are able to generate equilibria which are strictly Pareto-superior to any equilibrium of the game without a threshold contract. We shall call this feature *Pareto-improvability of* $\Gamma(-\infty)$.

As a second condition, we require that the support of the noise distribution is connected. Pareto-improvability of $\Gamma(-\infty)$ ensures that the welfare-maximizing threshold equilibrium involves a reappointment probability strictly below one. Connectedness of the support then guarantees that the principal will not raise the reappointment threshold beyond what is required by the contract, since doing so would increase the probability of deselecting the agent, which he wants to avoid if the agent's effort is higher than w_*.

To shape the first condition, we define:

Definition 6. *(i) A threshold equilibrium $(e, p) \in \mathcal{T}$ is called a* proper contract equilibrium *if (e, p) is not an equilibrium of $\Gamma(-\infty)$.*

(ii) We say that a threshold equilibrium $(e, p) \in \mathcal{T}$ is a Pareto-improvement *over $\Gamma(-\infty)$ if (e, p) strictly dominates every threshold equilibrium of $\Gamma(-\infty)$ in the Pareto sense—formally, if for each threshold equilibrium (e', p') of $\Gamma(-\infty)$ the inequalities $\mathfrak{v}(e, p) \geq \mathfrak{v}(e', p')$ and $\mathfrak{w}(e, p) \geq \mathfrak{w}(e', p')$ hold and at least one of the two inequalities is strict. If a Pareto-improvement exists, we call $\Gamma(-\infty)$* Pareto-improvable*.*

We collect some simple properties, which we will make use of later:

Remark 1. *(i) A threshold equilibrium $(e, p) \in \mathcal{T}$ can be a proper contract equilibrium only if $e > w_*$ and $\mathfrak{q}(e, p) \in (0; 1)$.*

(ii) If an equilibrium is a Pareto-improvement, it must be a proper contract equilibrium.

(iii) If $\Gamma(-\infty)$ is Pareto-improvable, then necessarily $w_* > \underline{e}$.

(iv) If $\Gamma(-\infty)$ is Pareto-improvable, then $\mathfrak{q}(e, p) \in (0; 1)$ for all $(e, p) \in \mathcal{T}$ with $e \geq w_*$.

(v) If $\Gamma(-\infty)$ is Pareto-improvable, then a proper contract equilibrium $(\hat{e}, \hat{p})$ exists that is welfare-maximizing in the sense of Corollary 5, that is, for which $\mathfrak{H}(\hat{e}, \hat{p}) = \max\{\mathfrak{H}(e, p) \mid (e, p) \in \mathcal{T}\}$.

Proof. Part (i): From Proposition 6 it immediately follows that every $(e, p) \in \mathcal{T}$ with $e \leq w_*$ or $\mathfrak{q}(e, p) \in \{0, 1\}$ is an equilibrium of $\Gamma(-\infty)$.

Part (ii) is trivial.

Part (iii): Suppose $w_* \leq \underline{e}$. Then, by Proposition 6, the pair $(\underline{e}, 1)$ is an equilibrium of $\Gamma(-\infty)$. Now consider any pair (e, p). If $e > \underline{e}$, then $\mathfrak{v}(e, p) < \mathfrak{v}(\underline{e}, 1)$. If $e = \underline{e}$, then $\mathfrak{v}(e, p) \leq \mathfrak{v}(\underline{e}, 1)$ and $\mathfrak{w}(e, p) \leq \mathfrak{w}(\underline{e}, 1)$. Hence (e, p) does not strictly dominate $(\underline{e}, 1)$.

Part (iv): Let (e^*, p^*) be a Pareto-improvement of $\Gamma(-\infty)$. Consider $(e, p) \in \mathcal{T}$ with $e \geq w_*$. Since $e > \underline{e}$, the agent has to be rewarded for his effort with a strictly positive probability of reappointment. If we had $\mathfrak{q}(e, p) = 1$, then (e, p) would be an equilibrium of $\Gamma(-\infty)$, by Proposition 6. Since, by Parts (i) and (ii), $\mathfrak{q}(e^*, p^*) < 1$, it follows that $e^* < e$, as otherwise we would have $\mathfrak{v}(e, p) > \mathfrak{v}(e^*, p^*)$, contradicting the fact that (e^*, p^*) is a Pareto-improvement. The inequalities $e^* < e$ and $\mathfrak{q}(e^*, p^*) < \mathfrak{q}(e, p)$, however, imply $\mathfrak{w}(e^*, p^*) < \mathfrak{w}(e, p)$, which is again a contradiction.

Part (v): By Corollary 5, a welfare-maximizing equilibrium $(\hat{e}, \hat{p})$ exists. Suppose $(\hat{e}, \hat{p})$ is *not* a proper contract equilibrium. Then, $(\hat{e}, \hat{p})$ is an equilibrium of $\Gamma(-\infty)$. Since $\Gamma(-\infty)$ is Pareto-improvable, some $(\hat{e}', \hat{p}') \in \mathcal{T}$ exists such that $\mathfrak{v}(\hat{e}', \hat{p}') \geq \mathfrak{v}(\hat{e}, \hat{p})$ and $\mathfrak{w}(\hat{e}', \hat{p}') \geq \mathfrak{w}(\hat{e}, \hat{p})$. From the monotonicity of the welfare function in the individual utilities it follows that $\mathfrak{H}(\hat{e}', \hat{p}') \geq \mathfrak{H}(\hat{e}, \hat{p})$; hence $(\hat{e}', \hat{p}')$ is a proper contract equilibrium that is welfare-maximizing. □

By the reasoning outlined above, we obtain the following proposition, which is the key step towards the desired results:

Proposition 7. *Suppose that $\Gamma(-\infty)$ is Pareto-improvable. Suppose, further, that the support of the noise distribution is connected. Consider a threshold τ. If $\Gamma(\tau)$ possesses a threshold equilibrium that is a proper contract equilibrium, then the threshold equilibria of $\Gamma(\tau)$ can be ordered in the Pareto sense. The ordering is strict for all threshold equilibria involving an effort level of at least w_*.*

A formal proof of the proposition is given in Appendix A.2. This proposition permits a corollary which states the existence of a threshold contract whose associated game has a unique welfare-maximizing threshold equilibrium that strictly dominates all other threshold equilibria of the game. The corollary is the first main result of this section:

Corollary 6. *Suppose that $\Gamma(-\infty)$ is Pareto-improvable. Suppose, further, that the support of the noise distribution is connected. Then a threshold $\hat{\tau} \in (-\infty; +\infty)$ and a threshold equilibrium $(\hat{e}, \hat{p})$ of $\Gamma(\hat{\tau})$ exist such that*

(i) $(\hat{e}, \hat{p})$ is welfare-maximizing, i. e.

$$\mathfrak{H}(e, p) \leq \mathfrak{H}(\hat{e}, \hat{p}) \text{ for all } (e, p) \in \mathcal{T}, \tag{14}$$

(ii) $(\hat{e}, \hat{p})$ strictly Pareto-dominates all other threshold equilibria of $\Gamma(\hat{\tau})$.

Proof. Let $\Gamma(-\infty)$ be Pareto-improvable and let the support of the noise distribution be connected. By Part (v) of Remark 1, a proper contract equilibrium $(\hat{e}, \hat{p})$ satisfying (14) exists. A threshold $\hat{\tau}$ exists such that $(\hat{e}, \hat{p})$ is a threshold equilibrium of $\Gamma(\hat{\tau})$. By Proposition 7, the threshold equilibria of $\Gamma(\hat{\tau})$ can be ordered in the Pareto sense, and the ordering is strict for effort levels strictly larger than $\underline{e}$. Hence, without loss of generality, $(\hat{e}, \hat{p})$ strictly dominates all threshold equilibria of $\Gamma(\hat{\tau})$. □

At the cost of an arbitrarily small loss in welfare, we can strengthen the preceding Corollary into a uniqueness result, which we shall give in Corollary 7. Corollary 4 tells us that the agent's best response is "generically" unique. By reducing $\hat{\tau}$ by an arbitrarily small amount, one can ensure that the resulting game possesses a unique equilibrium—apart from equilibria involving effort levels of $\underline{e}$ or w_*. As Corollary 1 suggests, the bad equilibrium at $\underline{e}$ cannot be avoided. Further, it is not hard to construct examples with lots of equilibria at w_*. Uniqueness up to $\underline{e}$ and w_* is, therefore, the best we can hope to achieve.

Heading for this result, we start by demonstrating that the welfare loss induced by a reduction of the threshold can be controlled. For each b', define

$$\overline{\mathcal{R}}(b') := \big\{e' \in \mathcal{R}(b') \bigm| e' > \max\{\underline{e}, w_*\}\big\}.$$

Proposition 8. *Consider some welfare function H, some $b \in (-\infty; +\infty)$, some $e \in \overline{\mathcal{R}}(b)$, and some $\varepsilon > 0$. Then, $\delta > 0$ exists such that for each $b' \in (b - \delta; b)$, one has $\overline{\mathcal{R}}(b') \neq \emptyset$ and*

$$\mathfrak{H}(e', \mathbb{1}_{[b';\infty)}) > \mathfrak{H}(e, \mathbb{1}_{[b;\infty)}) - \varepsilon \textit{ for all } e' \in \overline{\mathcal{R}}(b').$$

A proof of this proposition is given in Appendix A.2. The proposition leads the path to what we want to achieve. Let $(\hat{e}, \hat{p})$, with $\hat{p} = \mathbb{1}_{[\hat{b};\infty)}$, be a proper contract equilibrium. Then $(\hat{e}, \hat{p})$ is an equilibrium of the game $\Gamma(\hat{\tau})$, with $\hat{\tau} := \hat{b}$. Now we can reduce $\hat{\tau}$ by an arbitrarily small amount to, say, τ. The generic uniqueness of the best response ensures that this can be done in a way that the best response e to $\mathbb{1}_{[\tau;\infty)}$ is unique, and Proposition 8 guarantees that the welfare resulting from the equilibrium $(e, \mathbb{1}_{[\tau;\infty)})$ is only slightly below the maximum welfare resulting from $(\hat{e}, \mathbb{1}_{[\hat{\tau};\infty)})$. This is formalized in the following corollary, the proof of which is given in Appendix A.2. Observe that, by Part (v) of Remark 1, a welfare-maximizing equilibrium is, without loss of generality, a proper contract equilibrium; thus the corollary indeed applies to any welfare-maximizing equilibrium $(\hat{e}, \hat{p})$ if $\Gamma(-\infty)$ is Pareto-improvable. It is the second main result of this section.

Corollary 7. *Suppose that* $\Gamma(-\infty)$ *is Pareto-improvable. Suppose, further, that the support of the noise distribution is connected. Let* $(\hat{e}, \hat{p}) \in \mathcal{T}$ *be a proper contract equilbrium. Then, for all* $\varepsilon > 0$*, some threshold* $\tau \in [-\infty; +\infty]$ *exists such that*

(i) the game $\Gamma(\tau)$ *possesses a threshold equilibrium* (e^*, p^*) *satisfying the inequality*

$$\mathfrak{H}(e^*, p^*) \geq \mathfrak{H}(\hat{e}, \hat{p}) - \varepsilon, \tag{15}$$

(ii) (e^*, p^*) *strictly Pareto-dominates all other threshold equilibria of* $\Gamma(\tau)$*, and*

(iii) (e^*, p^*) *is the only threshold equilibrium of* $\Gamma(\tau)$ *with* $e^* \notin \{\underline{e}, w_*\}$.

4.5. Maximizing the utility of the principal

In this section, we are going to examine further the case in which the welfare function depends only on its second argument. Then, welfare is determined entirely by the utility of principal, and our previous results turn into statements about threshold contracts maximizing this utility. For example, Corollary 5, when applied to the welfare function $H(x, y) = y$, states the existence of a threshold equilibrium in which the principal's utility is maximized. The analogue to Corollary 7 reads as follows:

Proposition 9. *Suppose the support of the noise distribution is connected. Then, for all* $\varepsilon > 0$*, there is some threshold* $\tau \in [-\infty; +\infty]$ *such that*

(i) the game $\Gamma(\tau)$ *possesses a threshold equilibrium* (e^*, p^*) *satisfying the inequality*

$$\mathfrak{w}(e^*, p^*) \geq \max\big\{\mathfrak{w}(e, p) \;\big|\; (e, p) \in \mathcal{T}\big\} - \varepsilon, \tag{16}$$

(ii) every threshold equilibrium (e^*, p^*) *of* $\Gamma(\tau)$ *with* $e^* > \max\{\underline{e}, w_*\}$ *satisfies Inequality* (16) *and strictly Pareto-dominates all threshold equilibria of* $\Gamma(\tau)$ *involving an effort level of* $\underline{e}$ *or* w_*.

The proof of this proposition is very similar to that of Corollary 7; it is given in Appendix A.2. The main difference between the two results is that Pareto-improvability of $\Gamma(-\infty)$ is no longer required. In Corollary 7, it was needed to ensure that the reappointment probability in equilibrium was strictly below one, which, together with the connectedness of the support, ensured that the principal does not raise the threshold beyond τ, which would give room for other, welfare-inferior equilibria. This is not a problem in Proposition 9. If, on the one hand, the reappointment probability in equilibrium is below one, the arguments used in the proof of Corollary 7 still apply. If, on the other hand, reappointment probability in equilibrium is one, the principal's using a higher threshold does not harm, as any equilibrium involving a higher threshold (and an effort level beyond $\max\{\underline{e}, w_*\}$) will still imply a reappointment probability of one, yet a higher effort of the agent, and thus yield even higher utility for the principal.

In Section 5.3, we will demonstrate that resorting to equilibria that *approximate* a welfare optimum (Section 4.4) or the maximal utility of the principal (Section 4.5) is indeed necessary to guarantee uniqueness (up to equilibria involving $\underline{e}$ or w_*). Otherwise multiple equilibria or even a continuum of equilibria associated with a particular threshold may emerge.

4.6. Maximizing the utility of the agent

Having examined utility maximization of the principal, we now take the opposite view. It turns out, however, that threshold contracts are of no help in establishing the equilibrium that would be best for the agent, due to their asymmetric nature, in that they prevent, yet do not enforce reappointment. The following proposition summarizes the result:

Proposition 10. *A threshold equilibrium exists in which the agent's utility is maximal, i. e.* $(\hat{e}, \hat{p}) \in \mathcal{T}$ *exists such that* $\mathfrak{v}(\hat{e}, \hat{p}) = \max\{\mathfrak{v}(e, p) \mid (e, p) \in \mathcal{T}\}$*. Necessarily,* $\hat{e} \in \{\underline{e}, w_*\}$*.*

Proof. The existence of $(\hat{e}, \hat{p})$ follows from Corollary 5 with $\mathfrak{H} \equiv \mathfrak{v}$. If $w_* < \underline{e}$, then the statement is trivial, since $(\underline{e}, 1)$ is a $\mathfrak{v}$-maximizing equilibrium. Suppose $\hat{e} > w_* \geq \underline{e}$. The strategy $\hat{p}$ is a threshold strategy, hence $\hat{p} = \mathbb{1}_{[\hat{b};\infty)}$ for some $\hat{b}$. By Part (v) of Lemma 2, some threshold $b' \leq \hat{b}$ exists such that $w_* \in \mathcal{R}(b')$. By Proposition 6, the pair $(w_*, \mathbb{1}_{[b';\infty)})$ is an equilibrium of all games $\Gamma(\tau)$ with $\tau \leq b'$; thus $(w_*, \mathbb{1}_{[b';\infty)}) \in \mathcal{T}$. By Part (iv) of Lemma 2, $\mathfrak{q}(w_*, \mathbb{1}_{[b';\infty)}) \geq \mathfrak{q}(\hat{e}, \hat{p})$. Thus, $\mathfrak{v}(w_*, \mathbb{1}_{[b';\infty)}) > \mathfrak{v}(\hat{e}, \hat{p})$, which is a contradiction. ☐

As mentioned in the proof, the $\mathfrak{v}$-maximizing equilibrium $(\hat{e}, \hat{p})$ is an equilibrium under any threshold contract that does not rule out the strategy $\hat{p}$. In particular, it is an equilibrium of the game $\Gamma(-\infty)$, in which no minimum threshold is fixed. Since, generally, the principal is not prohibited from pursuing a strategy involving a threshold higher than fixed in the contract, a threshold contract cannot guarantee to the agent that the equilibrium optimal for him will be played.

The reader should, however, note that all this does *not necessarily* imply that the agent will not be interested in establishing a threshold contract. When, for instance, several candidates compete for office, candidates can gain credibility by offering threshold contracts. In addition, as threshold contracts may enlarge the set of possible equilibria (an example will be given in Section 5.1), they increase the scope for negotiation.

5. Examples

In this section we illustrate our findings by several examples. The key tool for analysing equilibria under threshold contracts is Proposition 6. In Section 5.1 we look at the case of exponentially distributed noise, demonstrating how threshold contracts yield Pareto-superior outcomes. Section 5.2 then deals again with uniformly distributed noise. Section 5.3 illustrates Corollary 7 and Proposition 9 by showing that one has to permit a small reduction in welfare or utility if the number of equilibria is supposed to be small.

5.1. Example: Exponential distribution

Let noise follow a shifted and mirrored exponential distribution. Specifically, we assume that for some $\lambda > 0$ the expression $-A + 1/\lambda$ is exponentially distributed with the parameter λ. The term $1/\lambda$ is the expected value of an exponential distribution with parameter λ; hence $\mathrm{E}[A] = 0$. The cumulative distribution function of $-A$ reads

$$F_{-A}(x) = \begin{cases} 1 - \exp(-\lambda x - 1) & \text{for } x > -1/\lambda, \\ 0 & \text{for } x \leq -1/\lambda. \end{cases} \tag{17}$$

To avoid distinguishing left-hand and right-hand derivatives, we assume that v is continuously differentiable.

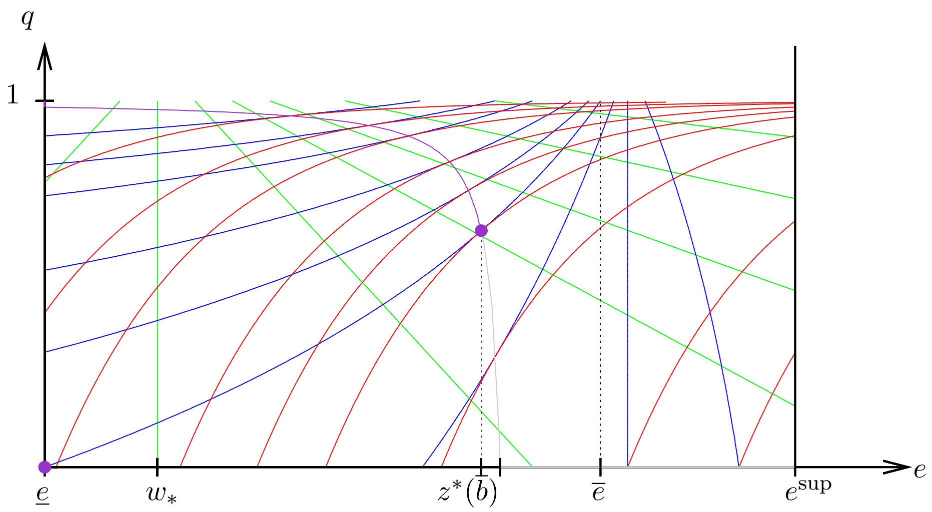

Figure 3: This figure illustrates the example from Section 5.1. Like the previous figures, it shows the indifference curves of the principal (green) and the agent (blue). The noise follows a shifted and mirrored exponential distribution. The red lines depict the functions $e \mapsto \mathfrak{q}(e, \mathbb{1}_{[b;\infty)})$, for different values of b. The violet curve, together with the point $(\underline{e}, 0)$ and the violet-coloured small segment on the vertical axis, is the set of points $\big(e, \mathfrak{q}(e, \mathbb{1}_{[b;\infty)})\big)$ for which $e \in \mathcal{R}(b)$. The points on the grey curve are "local" best responses (best responses within the concave part of the corresponding red curve), but are, from the agent's point of view, strictly worse than point $(\underline{e}, 0)$.

Figure 3 gives a graphical illustration. The functions $e \mapsto \mathfrak{q}(e, \mathbb{1}_{[b;\infty)})$ for different values of b are represented by the red curves. Since $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = F_{-A}(e-b)$, they are generated by shifting the graph of the cumulative distribution function F_{-A} horizontally to the right by b units. As we have already seen in Section 3.2, the agent's best response is the effort level at which the red curve hits the highest indifference curve (blue line) of the agent. Since the F_{-A} is concave in the segment where it takes strictly positive values, and since the agent's indifference curves are convex, there exists a unique effort level within the concave part of the red line that is a best response for the agent, compared to all other effort levels in the concave part of the red line. We use $z^*(b)$ to denote this optimal effort level within the concave part—formally, $z^*(b)$ is the—uniquely defined—point from the interval $[-1/\lambda + b; e^{\text{sup}})$ satisfying the inequality

$$\mathfrak{v}\big(z^*(b), \mathbb{1}_{[b;\infty)}\big) \geq \mathfrak{v}\big(e, \mathbb{1}_{[b;\infty)}\big) \quad \text{for all } e \in [-1/\lambda + b; e^{\text{sup}}).$$

For $b \geq e^{\text{sup}} + 1/\lambda$ (i. e. if the concave part of the red curve starts to the right of e^{sup}), $z^*(b)$ does not exist. The geometric locus of the points

$$\Big(z^*(b), \mathfrak{q}\big(z^*(b), \mathbb{1}_{[b;\infty)}\big)\Big)$$

is given by the violet/grey-coloured curve (with the exception of the point $(\underline{e}, 0)$) in Figure 3.

The "local" best response $z^*(b)$, however, may yield a lower utility for the agent than the minimum effort level $\underline{e}$. Graphically, this is the case if the corresponding point in the diagram is below the blue indifference curve running through the point $(\underline{e}, 0)$. In the diagram, the "global" best responses of the agent are marked violet; the grey part of the curve indicates those local best responses that are strictly worse than $\underline{e}$.

We use $\bar{b}$ to denote that threshold level for which

$$\mathfrak{v}\big(z^*(\bar{b}), \mathbb{1}_{[\bar{b};\infty)}\big) = \mathfrak{v}(\underline{e}, 0).$$

Hence $z^*(\bar{b})$ is the effort level where the violet segment ends and the grey segment starts. With this notation, we can characterize the agent's best responses as follows:

$$\mathcal{R}(b) = \begin{cases} \big\{z^*(b)\big\} & \text{for } b < \bar{b}, \\ \big\{\underline{e}, z^*(\bar{b})\big\} & \text{for } b = \bar{b}, \\ \{\underline{e}\} & \text{for } b > \bar{b}. \end{cases}$$

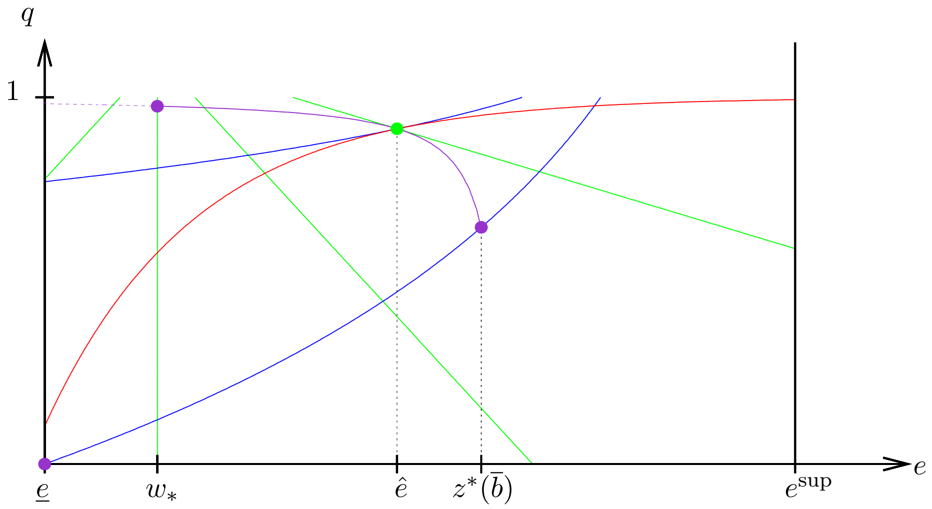

Figure 4: The points (e, p) on the violet curve with $e \in [w_*; z^*(\bar{b})]$, as well as the point $(\underline{e}, 0)$, can result as the equilibrium outcome under a threshold contract. At the points (e, p) on the violet curve with $e < w_*$ (drawn dashed violet), the agent plays a best response, but the principal would like to deviate to the strategy $\mathsf{0}$. The red curve and the green dot represent the equilibrium yielding the highest expected utility of the principal.

Having considered the agent, we now turn to the principal. For $w_* > \underline{e}$, the pair $\left(z^*(b), \mathbb{1}_{[b;\infty)}\right)$ does not constitute an equilibrium if $z^*(b) < w_*$ and the probability of reappointment is strictly larger than zero, since the strategy profile would then lead to an expected outcome for the principal that would be lower than his outside option, so the principal would like to deviate to a strategy involving no reappointment (for instance, to the strategy $\mathsf{0}$). In Figure 4, the corresponding segment of the violet curve is dashed. Hence, the set of threshold equilibrium outcomes (i. e. those effort level/reappointment probability pairs that can occur in a threshold equilibrium)

$$\left\{ \left(e, \mathfrak{q}(e, \mathbb{1}_{[b;\infty)})\right) \mid (e, \mathbb{1}_{[b;\infty)}) \in \mathcal{T}, b \in [-\infty; +\infty] \right\}$$

corresponds to the non-dashed segment of the violet curve, including its endpoints, as well as the point $(\underline{e}, 0)$.

Note that in all threshold equilibria involving an effort level strictly greater than $\underline{e}$, the reappointment probability is strictly below one. Hence a pair $\left(z^*(b), \mathbb{1}_{[b;\infty)}\right) \in \mathcal{T}$ with $z^*(b) > w_*$ constitutes an equilibrium of a game $\Gamma(\tau)$ if and only if $\tau = b$.

We use this example to illustrate the welfare and utility considerations from Sections 4.3–4.5.

We first consider maximization of the principal's utility. The existence of a threshold equilibrium yielding maximum utility for the principal is guaranteed by Corollary 5 with $\mathfrak{H} = \mathfrak{w}$. With the help of Figure 4, determining this equilibrium is easy. Its outcome is given by that point within the set of violet points which is located on the highest indifference curve of the principal. This point is represented by the green circle in Figure 4. It can be calculated numerically by deriving an equation for the violet curve and then determining that point on the curve which yields the highest utility for the principal. As the diagram indicates, the $\mathfrak{w}$-maximizing equilibrium is uninteresting if $w_* > z^*(\bar{b})$; then the associated outcome is $(\underline{e}, 0)$. If $w_* \leq z^*(\bar{b})$, the equilibrium is of the form $\left(\hat{e}, \mathbb{1}_{[\hat{\tau};\infty)}\right)$ with $\hat{e} = z^*(\hat{\tau})$, for some threshold $\hat{\tau} \in [-\infty; \bar{b}]$. It is an equilibrium of the game $\Gamma(\hat{\tau})$. For any $\tau \neq \hat{\tau}$, it is not an equilibrium of $\Gamma(\tau)$.

The location of the $\mathfrak{w}$-maximizing point on the violet curve depends on the principal's discount factor γ. For lower values of γ, the principal's indifference curves will be steeper, and the green point will move to the right. With a lower discount factor γ (i. e. a higher discount rate), the principal favours a higher effort level, at the cost of a higher risk of inadvertently deselecting the agent in the future.

We summarize our findings:

Fact. *Suppose $-A$ follows a shifted exponential distribution. If $w_* \leq z^*(\bar{b})$, a unique threshold $\hat{\tau}$ (i. e. a unique threshold contract) exists such that the associated game $\Gamma(\hat{\tau})$ possesses a threshold equilibrium yielding the maximum utility for the principal. The game may possess other threshold equilibria. In all these other equilibria, effort is $\underline{e}$ and the probability of reappointment is zero.*

Figure 4 presents a parameter constellation for which the outcome in the favourable equilibrium under a non-trivial threshold contract is strictly Pareto-superior to the equilibrium outcome without a threshold contract.

We generalize this example by considering the maximization of an additive welfare function: Let public welfare be given by a weighted sum of the players' utilities:

$$\mathfrak{H}(e, p) = \xi \mathfrak{v}(e, p) + (1 - \xi)\mathfrak{w}(e, p), \tag{18}$$

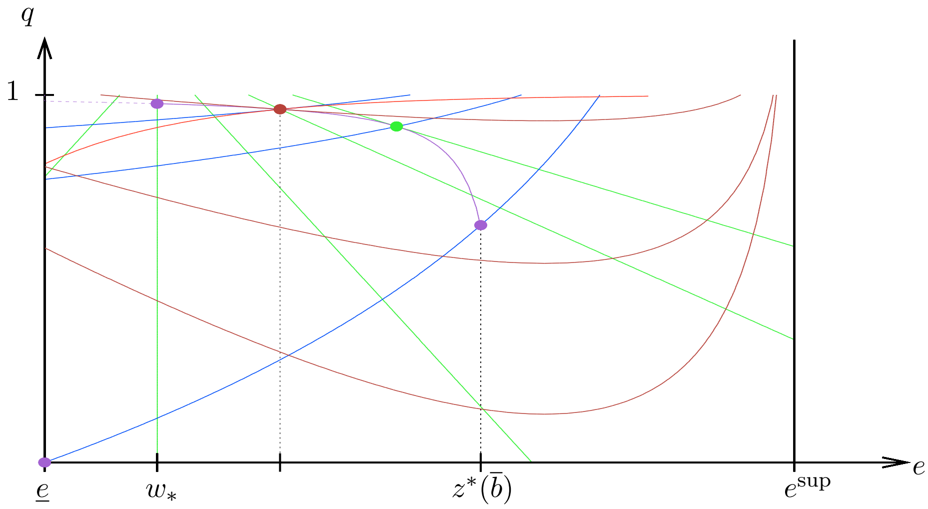

Figure 5: Similar set-up as in Figure 4. The brown curves represent contour lines of an additive welfare function where welfare is a weighted sum of both players' utilities. Higher curves represent higher welfare levels. The brown dot represents effort and reappointment probability in the welfare-maximizing equilibrium.

with the parameter $\xi \in [0; 1]$ describing the weight of the agent. Thus our analysis so far—maximization of the principal's utility—corresponds to welfare maximization with $\xi = 0$.

The brown curves in Figure 5 represent equipotential lines of $\mathfrak{H}$ for some $\xi \in (0; 1)$. The brown circle represents the outcome in the welfare-maximizing threshold equilibrium. Since, in this equilibrium, the reappointment probability is strictly below one, the equilibrium must be supported by a threshold contract. The threshold fixed in the contract is lower than the threshold required for the equilibrium in which the principal's utility is maximal (analysed above and represented by the green circles in Figures 4 and 5). The left end-point of the solid violet curve (represented by a violet circle and located at w_*) is welfare-maximizing for $\xi = 1$, thus being the equilibrium best for the agent.

If we let the parameter ξ, the agent's weight in the welfare function, run from 0 to 1, the welfare optimum moves, starting from the green circle, along the violet curve, until it reaches the left end point of the solid violet curve, the violet circle located at w_*. Hence the higher the weight of the agent, the lower is the welfare-optimal equilibrium effort level, and the higher is the probability of reappointment.

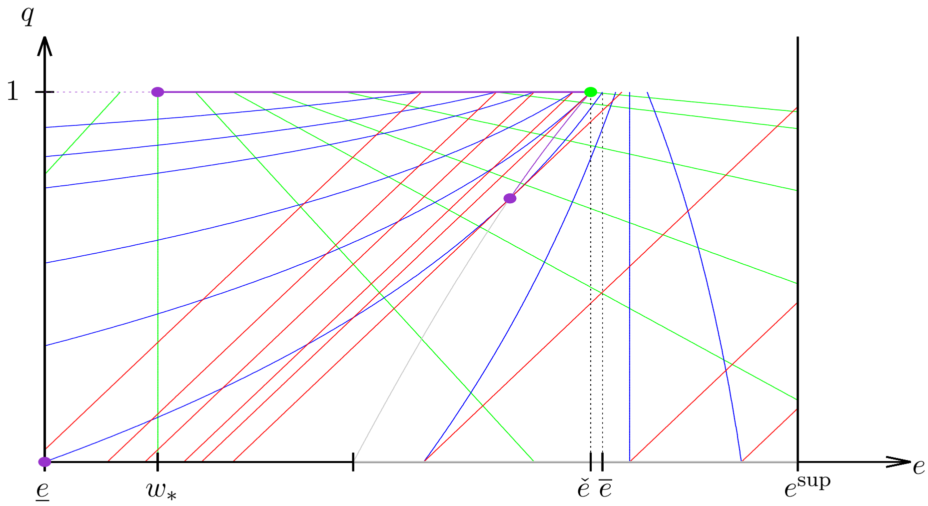

Figure 6: The example with uniform distribution of the noise discussed in Section 5.2.

5.2. Example: Uniform distribution (continued)

In this section we continue the example from Section 3.3 in which the noise is uniformly distributed. We analyse the equilibria under threshold contracts and determine the $\mathfrak{w}$-maximizing equilibrium.

For simplicity, we again assume that v is continuously differentiable. With $-A$ being uniformly distributed on the interval $[-\overline{a}; +\overline{a}]$, the cumulative distribution function is concave on the segment where it takes values strictly greater than zero; hence we can use similar arguments as in the previous example. Again, we have two candidates for the agent's best response—a "local" best response $z^*(b)$ in the concave part of the red curves as well as the minimum effort level $\underline{e}$.

As in the previous example, the violet/grey curve in Figure 6 represents the function $z^*(\cdot)$. As the diagram shows, there is a maximum effort level $\check{e}$ that can occur as a local best response; this effort level is given as that value of e^* for which Inequality (8) holds with equality. If $\mathsf{v}(\check{e}, 1) \geq \mathsf{v}(\underline{e}, 0)$, which we shall assume in the following, this local best response is not outreached by $\underline{e}$. With $\check{b} := \check{e} - \overline{a}$, the pair $(\check{e}, \mathbb{1}_{[\check{b};\infty)})$ is a threshold equilibrium of all games $\Gamma(b)$ with $b \in [-\infty; \check{b}]$. The reappointment probability in this equilibrium is one.

Further equilibria involving an effort level of at least w_*, however, may exist. For $\tau \leq \check{b}$, all pairs $(e, \mathbb{1}_{[b;\infty)})$ with $\max\{\tau, w_* - \overline{a}\} \leq b \leq \check{b}$ and $e = b + \overline{a}$ constitute an equilibrium

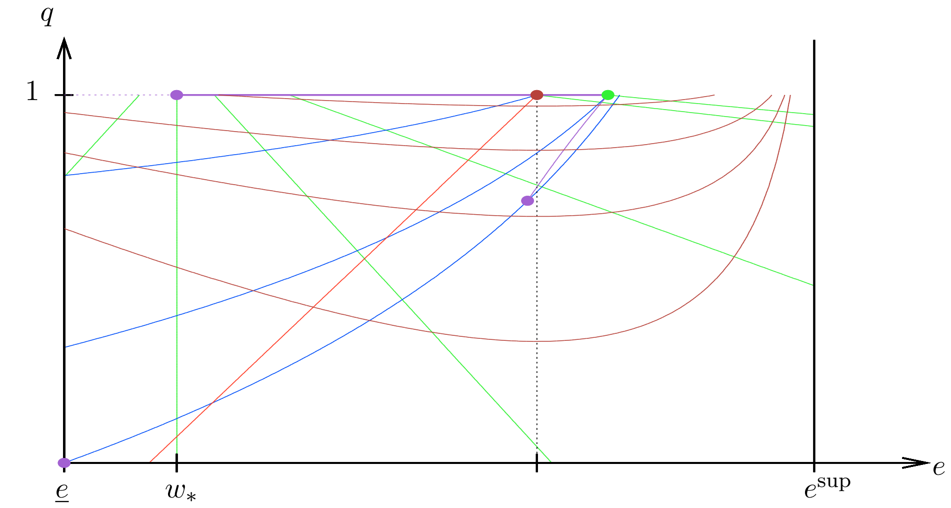

Figure 7: Welfare maximization for uniform noise distribution. The brown curves represent contour lines of a welfare function. The brown circle represents the outcome in the welfare-maximizing threshold equilibrium.

of $\Gamma(\tau)$. In all these equilibria, the reappointment probability is one.

For $b \geq \check{b}$, the "local" best response $z^*(b)$ is strictly decreasing in b. For $b > \check{b}$ with $\mathfrak{v}\big(z^*(b), \mathbb{1}_{[b;\infty)}\big) \geq \mathsf{v}(\underline{e}, 0)$, the pair $\big(z^*(b), \mathbb{1}_{[b;\infty)}\big)$ is a threshold equilibrium of the game $\Gamma(b)$. From the diagram one immediately observes that all these threshold equilibria are Pareto-dominated by $(\check{e}, \mathbb{1}_{[\check{b};\infty)})$. In particular, $\Gamma(-\infty)$ is not Pareto-improvable.

Hence, with respect to utility maximization for the principal, one can say the following: The $\mathfrak{w}$-maximizing threshold equilibrium is given by the strategy profile $(\check{e}, \mathbb{1}_{[\check{b},\infty)})$. It is an equilibrium of $\Gamma(\tau)$ if and only if $\tau \leq \check{b}$. Hence the $\mathfrak{w}$-maximizing threshold equilibrium can be reached without a threshold contract (in contrast to the previous example, in which noise followed a shifted exponential distribution). By installing a threshold contract of $\check{b}$, one can ensure that all threshold equilibria above $\max\{\overline{e}, w_*\}$ except the $\mathfrak{w}$-maximizing one disappear.

Again, we conclude the example by considering an additive welfare function. Like in the previous section, let welfare be given by Equation (18). Figure 7 illustrates such a situation. If both players have strictly positive weight (i. e. if $\xi \in (0;1)$), then the welfare-maximizing equilibrium outcome lies somewhere on the horizontal segment of the violet curve, with a reappointment probability of one and an effort level strictly between w_* and the effort level desired by the principal (green circle).

Since, generally, nothing prevents the principal from playing a strategy that is higher than the threshold fixed by the threshold contract and since, in this example, the equilibrium probability does not go down if the threshold is increased beyond the welfare-maximizing one—which would keep the principal from doing it—, there is no threshold contract that can ensure that the brown, rather than, for instance, the green equilibrium is played. The equilibrium outcomes located to the right of the brown circle are superior for the principal, but inferior for the agent. We observe that here a threshold contract cannot ensure a welfare-optimal equilibrium, even if we use dominance arguments. Thus the present example shows that Pareto-improvability of $\Gamma(-\infty)$ is indeed a necessary condition in Corollary 6.

5.3. A continuum of threshold equilibria

In this section, we present an example in which

(a) a unique threshold equilibrium $(\hat{e}, \hat{p})$ maximizing $\mathfrak{w}(e, p)$ among all $(e, p) \in \mathcal{T}$ exists,

(b) a unique threshold $\hat{\tau}$ exists such that $(\hat{e}, \hat{p})$ is a threshold equilibrium of $\Gamma(\hat{\tau})$,

(c) $\mathcal{R}(\hat{\tau}) = \{\underline{e}\} \cup [w_*; \hat{e}]$, and

(d) the threshold equilibria of $\Gamma(\hat{\tau})$ cover all utility levels of the principal that can be achieved in any threshold equilibrium, i. e.

$$\{\mathfrak{w}(e, p) \mid (e, p) \text{ is a threshold equilibrium of } \Gamma(\hat{\tau})\} = \{\mathfrak{w}(e, p) \mid (e, p) \in \mathcal{T}\}.$$

The game $\Gamma(\hat{\tau})$ possesses a continuum of threshold equilibria. Hence, if a threshold contract fixing $\hat{\tau}$ is established, we need to resort to dominance arguments to justify why the equilibrium $(\hat{e}, \hat{p})$, of all equilibria, should be played. This is an "extreme" example showing that $\Gamma(\hat{\tau})$ may possess many other threshold equilibria besides the $\mathfrak{w}$-maximizing equilibrium $(\hat{e}, \hat{p})$, and that thus one cannot set $\varepsilon = 0$ in Corollary 7 or Proposition 9.

The example is illustrated in Figure 8. The noise distribution is constructed as follows: We consider a random variable X whose cumulative distribution function F_X is given

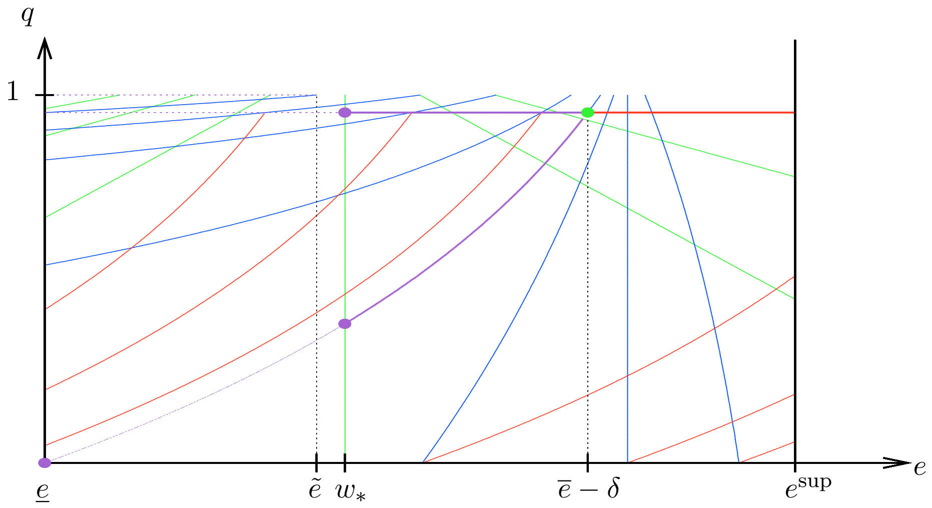

Figure 8: This figure illustrates the example from Section 5.3, in which $\Gamma(\hat{\tau})$ possesses a continuum of threshold equilibria.

by

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq \underline{e}, \\ \beta^{-1} \cdot \dfrac{v(\underline{e}) - v(x)}{v(\underline{e}) - v_*} & \text{for } \underline{e} < x < \overline{e} - \delta, \\ \beta^{-1} \cdot \dfrac{v(\underline{e}) - v(\overline{e} - \delta)}{v(\underline{e}) - v_*} & \text{for } \overline{e} - \delta \leq x < 2\overline{e} - \underline{e}, \\ 1 & \text{for } x \geq 2\overline{e} - \underline{e}, \end{cases}$$

with $\delta \in (0; \overline{e} - \underline{e})$ being some constant. By construction, for $e \in (\underline{e}; \overline{e} - \delta)$ the function F_X fulfils the identity

$$\mathsf{v}\big(e, F_X(e)\big) = \mathsf{v}(\underline{e}, 0),$$

i. e. on the interval $[\underline{e}; \overline{e} - \delta]$ the graph of F_X coincides with the agent's indifference curve running through the point $(\underline{e}, 0)$. On the interval $[\overline{e} - \delta; 2\overline{e} - \underline{e})$, the function F_X is constant at a level strictly below 1. At $2\overline{e} - \underline{e}$, it jumps to 1.

Let $\tilde{e}$ be defined by the equation

$$\mathsf{v}\big(\tilde{e}, 1\big) = \mathsf{v}\big(\underline{e}, F_X(\overline{e} - \delta)\big).$$

It is straightforward to prove that $\tilde{e}$ exists, that it is unique, and that $\underline{e} < \tilde{e} < \overline{e}$. If δ is small, then $\tilde{e}$ is near $\underline{e}$.

With $\eta := \mathrm{E}[X]$, let $-A$ be distributed like $X - \eta$, i.e. the distribution of X is shifted such that the expected value is zero. Obviously, $\eta \in (\underline{e}; \overline{e})$. It is easy to verify that $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = 1$ if and only if $e \geq b - \eta + 2\overline{e} - \underline{e}$. Further, $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = F_X(\overline{e} - \delta)$ if and only if $b - \eta + \overline{e} - \delta \leq e < b - \eta + 2\overline{e} - \underline{e}$. Using this, we can determine the agent's best responses from Figure 8:

$$\mathcal{R}(b) = \begin{cases} \{\underline{e}\} & \text{for } b \leq 2(\underline{e} - \overline{e}) + \eta, \\ \{b - \eta + 2\overline{e} - \underline{e}\} & \text{for } 2(\underline{e} - \overline{e}) + \eta < b < \tilde{e} + \underline{e} - 2\overline{e} + \eta, \\ \{\underline{e}, \tilde{e}\} & \text{for } b = \tilde{e} + \underline{e} - 2\overline{e} + \eta, \\ \{\underline{e}\} & \text{for } \tilde{e} + \underline{e} - 2\overline{e} + \eta < b \leq \underline{e} - \overline{e} + \delta + \eta, \\ \{b - \eta + \overline{e} - \delta\} & \text{for } \underline{e} - \overline{e} + \delta + \eta < b < \eta, \\ [\underline{e}; \overline{e} - \delta] & \text{for } b = \eta, \\ \{\underline{e}\} & \text{for } b > \eta. \end{cases}$$

For low threshold values b, the agent's best response is $\underline{e}$; the associated probability of reappointment is 1. When b rises, the best response will ascend toward $\tilde{e}$ and then drop to $\underline{e}$ again. Simultaneously, the reappointment probability will drop from 1 to $F_X(\overline{e} - \delta)$. When b rises further, the best response first remains at $\underline{e}$, then it rises to $\overline{e} - \delta$, with the reappointment probability remaining constant at $F_X(\overline{e} - \delta)$. If $b = \eta$, there is a continuum of best responses $\mathcal{R}(\eta) = [\underline{e}; \overline{e} - \delta]$. Graphically, this is reflected by the red curve's coinciding with the blue indifference curve which runs through the point $(\underline{e}, 0)$. For thresholds $b > \eta$, the agent's best response is $\underline{e}$, and the associated probability of reappointment is 0.

From $\mathcal{R}(b)$ we can derive the threshold equilibria of the games $\Gamma(\tau)$. We restrict ourselves to the most interesting case, in which $w_* \in (\tilde{e}; \overline{e} - \delta)$. The threshold equilibria of $\Gamma(\tau)$ are given by the set

$$\begin{aligned} &\big\{(\underline{e}, \mathbb{1}_{[b;\infty)}) \bigm| b \geq \max\{\tau, \eta\}\big\} \\ &\qquad \cup \begin{cases} \big\{(w_*, \mathbb{1}_{[\eta;\infty)}),\ (\tau - \eta + \overline{e} - \delta, \mathbb{1}_{[\tau;\infty)})\big\} & \text{for } w_* - \overline{e} + \delta + \eta \leq \tau < \eta, \\ \big\{(e, \mathbb{1}_{[\eta;\infty)}) \bigm| e \in [w_*; \overline{e} - \delta]\big\} & \text{for } \tau = \eta, \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

From Figure 8 we observe that this example has the Properties (a)–(d) stated at the beginning of this section. The $\mathfrak{w}$-maximizing threshold equilibrium $(\hat{e}, \hat{p})$ is given by

$\hat{e} := \overline{e} - \delta$, $\hat{p} := \mathbb{1}_{[\hat{\tau};\infty)}$, and $\hat{\tau} := \eta$. The strategy profile $(\hat{e}, \hat{p})$ is an equilibrium of the game $\Gamma(\tau)$ if and only if $\tau = \hat{\tau}$. The game $\Gamma(\hat{\tau})$ has a continuum of threshold equilibria—each effort level from the set $\{\underline{e}\} \cup [w_*; \overline{e} - \delta]$ is attainable by a threshold strategy. The corresponding equilibrium outcomes lie on the agent's indifference curve running through point $(\underline{e}, 0)$.

For $\tau \in (w_* + \eta - \overline{e} + \delta; \hat{\tau})$, the game $\Gamma(\tau)$ possesses a unique threshold equilibrium that involves an effort level not in $\{\underline{e}, w_*\}$. For τ approaching $\hat{\tau}$ from below, the outcome in this equilibrium approaches the outcome of the $\mathfrak{w}$-maximizing equilibrium $(\hat{e}, \hat{p})$. This observation illustrates the approximation result from Proposition 9 and shows that one has to permit an arbitrarily small reduction in welfare if one wants to ensure uniqueness up to $\underline{e}$ and w_*. Moreover, the present example demonstrates that one cannot avoid the "bad" equilibria at $\underline{e}$ and w_* in Proposition 9. The reason is that for $\tau < \eta$ the strategy profiles $(\underline{e}, 0)$ and $(w_*, \mathbb{1}_{[\eta;\infty)})$ are equilibria of $\Gamma(\tau)$, and for $\tau > \eta$ each threshold equilibrium involves the effort level $\underline{e}$ and a reappointment probability of zero.

In the present example, the fact that $\delta > 0$ guarantees that for $\tau \neq \hat{\tau}$ the strategy profile $(\hat{e}, \hat{p})$ is not an equilibrium of $\Gamma(\tau)$, i. e. that the threshold contract yielding the $\mathfrak{w}$-maximizing equilibrium is unique. In the borderline case $\delta = 0$, the $\mathfrak{w}$-maximizing equilibrium is $(\overline{e}, \mathbb{1}_{[\eta;\infty)})$. This equilibrium involves a reappointment probability of one, and it is an equilibrium of $\Gamma(\tau)$ for all $\tau \leq \eta$. Hence, for $\delta = 0$, Properties (a), (c), and (d) on page 40 are fulfilled, while Property (b) is not.

6. Discussion

In this section we discuss various ways of embedding the reappointment game into larger games and other possible extensions.

6.1. Initial appointment

Let us briefly discuss how the initial appointment decision of the principal can be included in the reappointment game. For this purpose, consider the following extension of the game: Before the reappointment game starts, the principal decides whether to appoint the agent for period 1 or not, i. e., whether to play the reappointment game with the agent or not. We call the resulting game the *augmented game*.

The agent's strategies in the augmented game correspond to the agent's strategies in the reappointment game. The agent's actions, as described by a strategy $\mathbf{e}$, are now to be understood as conditional on the fact that the principal initially decided to appoint the agent. A strategy of the principal is a vector $(p_0, p_1, p_2, \dots)$, where $(p_1, p_2, \dots)$ is a strategy in the reappointment game, and $p_0 \in \{0; 1\}$ indicates whether the principal initially appoints the agent or not.

If the agent is not appointed initially, the agent's per-period utility will be v_*, and the per-period utility of the principal will be w_*. If the strategy profile $(\mathbf{e}, \mathbf{p})$ with $\mathbf{e} = (e_1, e_2, \dots)$ and $\mathbf{p} = (p_0, p_1, \dots)$ is played, the principal's expected utility amounts to

$$\mathfrak{w}_0(\mathbf{e}, \mathbf{p}) = \gamma p_0 \cdot \mathfrak{w}_1\big(\mathbf{e}, (p_1, p_2, \dots)\big) + (1 - p_0) \cdot \frac{\gamma}{1 - \gamma} w_*.$$

Note that we discount utility with an additional factor of γ, taking the perspective of the principal looking at the game in period 0, i. e. one period before the agent could produce output. Defining equilibrium for the augmented game is straightforward.

Definition 7. *A strategy profile $(\mathbf{e}, \mathbf{p})$ in the augmented game with strategies $\mathbf{e} = (e_1, \dots)$ and $\mathbf{p} = (p_0, p_1, \dots)$ is an* equilibrium *if $\big(\mathbf{e}, (p_1, \dots)\big)$ is an equilibrium in the reappointment game and the principal's initial appointment decision is a best response, i. e.*

$$\mathfrak{w}_0(\mathbf{e}, \mathbf{p}) \geq \mathfrak{w}_0(\mathbf{e}, \mathbf{p}') \quad \textit{for all } \mathbf{p}'. \tag{19}$$

Formally, we have simply adapted Definition 1 to the augmented game by requiring Condition (ii) of that definition to be satisfied for $t = 0$ as well. By reformulating Condition (19), we immediately achieve the following characterization of equilibrium in the augmented game:

Proposition 11. *A strategy profile $(\mathbf{e}, \mathbf{p})$ in the augmented game with $\mathbf{e} = (e_1, \dots)$ and $\mathbf{p} = (p_0, p_1, \dots)$ is an equilibrium if and only if it satisfies the following requirements:*

(i) The pair $\big(\mathbf{e}, (p_1, p_2, \dots)\big)$ is an equilibrium of the reappointment game.

(ii) If $\mathfrak{w}_1\big(\mathbf{e}, (p_1, p_2, \dots)\big) > (1 - \gamma)^{-1} w_*$, then $p_0 = 1$, and if $\mathfrak{w}_1\big(\mathbf{e}, (p_1, p_2, \dots)\big) < (1 - \gamma)^{-1} w_*$, then $p_0 = 0$. □

Condition (ii) is void if $\mathfrak{w}_1\big(\mathbf{e}, (p_1, p_2, \dots)\big) = (1-\gamma)^{-1} w_*$; in this case, p_0 can be arbitrary. The condition says that the principal will appoint the agent if the expected utility from appointing is higher than his outside option, and that he will not appoint the agent if the expected utility is lower than the outside option. In the case of stationary Markovian strategies e and p, the condition reduces to a condition on e:

Proposition 12. *Let (e, p) be a stationary Markovian equilibrium, and let $p_0 \in \{0, 1\}$. Then the strategy profile $\big((e, e, \dots), (p_0, p, p, \dots)\big)$ is an equilibrium of the augmented game if and only if (i) $p_0 = 0$ and $e \leq w_*$, or (ii) $p_0 = 1$ and $e \geq w_*$.* □

By definition, an equilibrium in the augmented game implies an equilibrium in the reappointment game. By Proposition 11, in turn, an equilibrium in the reappointment game generates an equilibrium in the augmented game by a suitable choice of p_0. Thus, equilibria in the reappointment game and in the augmented game correspond to each other in a very straightforward way, so that focusing on the reappointment game does not involve any loss of generality.

6.2. Endogenizing the outside option in stationary environments

In an environment in which the principal faces a pool of potential agents who are all identical, the principal's outside option w_* can be endogenized as being the payoff the principal expects to obtain when he appoints the next agent after deselecting the current one. This works for the reappointment game with and without threshold contracts. We outline the procedure for the game without threshold contracts.

Suppose all potential agents have the same preferences, and suppose the noise distribution is independent of the agent in office. We define the following correspondence:

$$\Xi\colon \mathbb{R} \twoheadrightarrow [\underline{e}; e^{\sup}),\ \ w_* \mapsto \big\{e \;\big|(e,p) \text{ is a stationary Markovian equilibrium} \\ \text{when the principal has the outside option } w_*\big\}.$$

We assume that the game the principal plays with the first appointed agent and all subsequently appointed agents is the same, except for the outside option of the principal. The latter will be determined as part of the equilibrium. This brings us to:

Definition 8. *A strategy profile $(e, p) \in \Sigma_{\mathrm{P}}(-\infty) \times \Sigma_{\mathrm{A}}$ is called a* stationary Markovian equilibrium with endogenous outside options *if (e, p) is an equilibrium in the game with the outside option $w_* = e$.*

This definition requires that the effort level in equilibrium is a fixed point of Ξ, i. e. that $e \in \Xi(e)$. We note that such equilibria with endogenous outside options always exist.

By Corollary 1, $(\underline{e}, 0)$ is an equilibrium with endogenous outside options. Our results can be used to determine whether other equilibria with outside options exist as well. As an example, we obtain the following variant of Proposition 3:

Proposition 13. *Let noise A be uniformly distributed on an interval $[-\overline{a}; \overline{a}] \subseteq \mathbb{R}$ for some $\overline{a} > 0$. Let $e^* \in (\underline{e}; e^{\text{indiff}})$. A stationary Markovian strategy p^* such that (e^*, p^*) is an equilibrium with outside options of the game $\Gamma(-\infty)$ exists if and only if $w_* \leq e^* \leq \overline{e}$ and*

$$-\frac{\partial^- v(e^*)}{v(e^*) - v_*} \leq \frac{\beta}{1-\beta} \cdot \frac{1}{2\overline{a}} \,. \tag{20}$$

p^* can be chosen as $p^*(w) = \mathbb{1}_{[e^*-\overline{a};\infty)}(w)$. □

6.3. Eliminating bad equilibria

As we saw in Section 4.4, Proposition 9, threshold contracts exist, say with τ^0, that almost maximize the payoff of the principal, but such contracts cannot avoid the bad equilibria in which the effort level is $\underline{e}$ or w_*. To ensure that agents propose τ^0, we can embed the augmented game in an even larger game in which in period -1 a pool of identical agents can offer threshold contracts. This will encourage agents to offer τ^0 when the principal prefers agents who offer τ^0 to any other threshold contract. Avoidance of the equilibrium $(\underline{e}, 0)$, however, is impossible without further consideration, even if the threshold contract τ^0 has been offered.

One approach would be to make deselection costly for the principal. The bad equilibrium vanishes if $w_* < \underline{e}$. Hence, a good design would involve the optimal threshold contract, while reducing the outside option of the principal. One case in which deselection is particularly expensive is when the agent makes investments in long-term projects. If these investments cannot be upheld by another agent—or can only be kept up at high additional cost—, deselection causes opportunity cost for the principal, which reduces the value of the principal's outside option. Accordingly, long-term projects generate an *incumbency advantage* for the agent in office, as e.g. discussed by Müller (2009, Chapter 5, pp. 94–122). By enforcing deselection of the agent in response to bad performance, threshold contracts can prevent the agent from exploiting this incumbency advantage. At the same time, the existence of an incumbency advantage may avoid equilibria of reciprocal distrust.

Another possibility for eliminating the equilibrium $(\underline{e}, 0)$ is the application of refinements to the equilibrium concept. In particular, Pareto-dominance will eliminate $(\underline{e}, 0)$. Usually coordination devices foster the application of refinements. The choice of τ^0 by the agent could be interpreted as such a coordination device on the set of equilibria that are not Pareto-dominated.

7. Conclusion

We have developed the characteristics of equilibria in infinitely repeated reappointment games. While threshold contracts are a device that can engineer Pareto-improvements, they cannot entirely prevent the predisposition of the principal–agent relationship to reciprocal distrust. Numerous issues deserve further scrutiny. We have detailed a couple of extensions and applications in the last section. Together with further conceivable extensions they constitute an entire research programme.

A. Proofs

A.1. Proofs for Section 3

Proof of Proposition 1. Let (e^*, p^*) be a pair of stationary Markovian strategies with $\mathfrak{v}_1(\tilde{\mathbf{e}}, p^*) > \mathfrak{v}_1(e^*, p^*)$ for some (not necessarily stationary Markovian) strategy $\tilde{\mathbf{e}} = (\tilde{e}_1, \tilde{e}_2, \dots)$ of the agent. We are going to prove that a stationary Markovian strategy e exists that satisfies $\mathfrak{v}_1(e, p^*) > \mathfrak{v}_1(e^*, p^*)$. As this is obvious if $\mathsf{v}(e^*, 1) < \mathsf{v}(\underline{e}, 0)$ (take $e = \underline{e}$), we can, without loss of generality, assume that $\mathsf{v}(e^*, 1) \geq \mathsf{v}(\underline{e}, 0)$.

For $m = 1, 2, \dots$ define the strategy $\tilde{\mathbf{e}}^m = (\tilde{e}_1^m, \tilde{e}_2^m, \dots)$ by

$$\tilde{e}_t^m = \begin{cases} \tilde{e}_t & \text{for } t < m, \\ e^* & \text{for } t \geq m. \end{cases}$$

For each m, the strategies $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{e}}^m$ do not differ in the periods $1, \dots, m-1$; hence, for any realization of the noise, the resulting utility levels $w_1, \dots, w_{m-1}$ in the first $m-1$ periods do not differ, regardless of whether the strategy profile $(\tilde{\mathbf{e}}, p^*)$ or the strategy profile $(\tilde{\mathbf{e}}^m, p^*)$ is played. Recall that the random variable

$$\delta_{1,m-1}(p^*, \dots, p^*; w_1, \dots, w_{m-1})$$

indicates whether the agent is in office in period m. Hence, the difference

$$\mathfrak{v}_1\big(\tilde{\mathbf{e}}^m, p^*\big) - \mathfrak{v}_1\big(\tilde{\mathbf{e}}, p^*\big)$$

equals the expected value of

$$\delta_{1,m-1}(p^*, \dots, p^*; w_1, \dots, w_{m-1}) \cdot \beta^{m-1} \cdot \Big(\mathfrak{v}_m(\tilde{\mathbf{e}}^m, p^*; w_0, \dots, w_{m-1}\big) - \mathfrak{v}_m\big(\tilde{\mathbf{e}}, p^*; w_1, \dots, w_{m-1}\big)\Big). \tag{21}$$

Since

$$\mathfrak{v}_m\big(\tilde{\mathbf{e}}, p^*; w_1, \dots, w_{m-1}\big) \leq \frac{v(\underline{e})}{1-\beta}$$

and since, by $\mathsf{v}(e^*, 1) \geq \mathsf{v}(\underline{e}, 0)$, one has

$$\mathfrak{v}_m\big(\tilde{\mathbf{e}}^m, p^*; w_1, \dots, w_{m-1}\big) = \mathfrak{v}_m\big(e^*, p^*; w_1, \dots, w_{m-1}\big) \geq \frac{v_*}{1-\beta}\,,$$

Expression (21) is, with probability one, larger than or equal to

$$-\beta^{m-1} \cdot \frac{v(\underline{e}) - v_*}{1-\beta}\,.$$

Since this term converges to zero for m approaching infinity, we can conclude that

$$\liminf_{m\to\infty} \Big(\mathfrak{v}_1\big(\tilde{\mathbf{e}}^m, p^*\big) - \mathfrak{v}_1\big(\tilde{\mathbf{e}}, p^*\big)\Big) \geq 0.$$

From this, and from the assumption that $\mathfrak{v}_1(\tilde{\mathbf{e}}, p^*) > \mathfrak{v}_1(e^*, p^*)$, it follows that some $m \in \mathbb{N}^*$ exists such that $\mathfrak{v}_1(\tilde{\mathbf{e}}^m, p^*) > \mathfrak{v}_1(e^*, p^*)$. Let M be the smallest of these m, which means

$$\mathfrak{v}_1(\tilde{\mathbf{e}}^m, p^*) \leq \mathfrak{v}_1(e^*, p^*) \text{ for } m < M \tag{22}$$

and

$$\mathfrak{v}_1(\tilde{\mathbf{e}}^M, p^*) > \mathfrak{v}_1(e^*, p^*). \tag{23}$$

Since we have $\tilde{\mathbf{e}}^1 = (e^*, e^*, \dots)$, it holds that $M \geq 2$. The inequalities (22) and (23) imply

$$\mathfrak{v}_1(\tilde{\mathbf{e}}^M, p^*) - \mathfrak{v}_1(\tilde{\mathbf{e}}^{M-1}, p^*) > 0. \tag{24}$$

The left-hand side of this inequality is the expected value of the expression

$$\begin{aligned}&\delta_{1,M-2}(p^*, \dots, p^*; w_1, \dots, w_{M-2}) \cdot \beta^{M-2}\\ &\quad\cdot \Big(\mathfrak{v}_{M-1}\big(\tilde{\mathbf{e}}^{M-1}, p^*; w_1, \dots, w_{M-2}\big) - \mathfrak{v}_{M-1}\big(\tilde{\mathbf{e}}^{M}, p^*; w_1, \dots, w_{M-2}\big)\Big).\end{aligned} \tag{25}$$

Let the random variable

$$\varepsilon := \tilde{e}_{M-1}(w_1, \dots, w_{M-2})$$

describe the agent's effort level in period $M-1$ if the strategy profile $(\tilde{\mathbf{e}}^M, p^*)$ is played. Now, expression (25) can be rewritten as

$$\delta_{1,M-2}(p^*, \dots, p^*; w_1, \dots, w_{M-2}) \cdot \beta^{M-2} \cdot \Big(\mathfrak{v}_1\big((\varepsilon, e^*, e^*, \dots), p^*\big) - \mathfrak{v}_1(e^*, p^*)\Big).$$

Inequality (24) implies that a value of ε must exist for which this expression is strictly positive; hence an effort level e exists such that

$$\mathfrak{v}_1\big((e, e^*, e^*, \dots), p^*\big) > \mathfrak{v}_1(e^*, p^*). \tag{26}$$

For $m \in \mathbb{N}^*$ define the strategy $\mathbf{e}^m = (e_0^m, e_1^m, \dots)$ by

$$e_t^m = \begin{cases} e & \text{for } t < m, \\ e^* & \text{for } t \geq m. \end{cases}$$

With this notation, (26) reads

$$\mathfrak{v}_1(\mathbf{e}^2, p) > \mathfrak{v}_1(\mathbf{e}^1, p).$$

By applying this to the identity

$$\mathfrak{v}_1(\mathbf{e}^m, p^*) + (1-\beta)^{-1} v_* = v(e) + \beta \mathfrak{q}(e, p^*) \big(\mathfrak{v}_1(\mathbf{e}^{m-1}, p^*) + (1-\beta)^{-1} v_* \big)$$

for $m = 2$ and $m = 3$, we reach

$$\begin{aligned} \mathfrak{v}_1(\mathbf{e}^3, p^*) + (1-\beta)^{-1} v_* &= v(e) + \beta \mathfrak{q}(e, p^*) \big(\mathfrak{v}_1(\mathbf{e}^2, p^*) + (1-\beta)^{-1} v_* \big) \\ &> v(e) + \beta \mathfrak{q}(e, p^*) \big(\mathfrak{v}_1(\mathbf{e}^1, p^*) + (1-\beta)^{-1} v_* \big) \\ &= \mathfrak{v}_1(\mathbf{e}^2, p^*) + (1-\beta)^{-1} v_*. \end{aligned}$$

Proceeding by induction, we achieve

$$\mathfrak{v}_1(\mathbf{e}^m, p^*) > \mathfrak{v}_1(\mathbf{e}^{m-1}, p^*) \text{ for all } m \geq 1,$$

which means that $\mathfrak{v}_1(\mathbf{e}^m, p^*)$ is monotonically increasing in m. We have

$$\big| \mathfrak{v}_1(\mathbf{e}^m, p^*) - \mathfrak{v}_1(e, p^*) \big| \leq \beta^m \cdot (1-\beta)^{-1} \cdot \left(\max_{p \in [0;1]} \big| \mathsf{v}(e, p) \big| + \max_{p \in [0;1]} \big| \mathsf{v}(e^*, p) \big| \right),$$

which implies $\mathfrak{v}_1(\mathbf{e}^m, p^*) \to \mathfrak{v}_1(e, p^*)$ for $m \to \infty$. From this and from the monotonicity of $\mathfrak{v}_1(\mathbf{e}^m, p^*)$ in m, it follows that

$$\mathfrak{v}_1(e, p^*) > \mathfrak{v}_1(e^*, p^*),$$

as we intended to show.

Now let (e^*, p^*) be a pair of stationary Markovian strategies with $\mathfrak{w}(e^*, \tilde{\mathbf{p}}) > \mathfrak{w}(e^*, p^*)$ for some strategy $\tilde{\mathbf{p}}$ of the principal. This implies $e^* \neq w_*$, since for $e^* = w_*$ the principal's expected utility amounts to $\mathfrak{w}(e^*, \mathbf{p}') = w_*/(1-\gamma)$ for all strategies $\mathbf{p}'$ and hence is the same for all strategies of the principal.

If $e^* > w_*$, one has $\mathfrak{w}(e^*, 1) \geq \mathfrak{w}(e^*, \mathbf{p}')$ for all strategies $\mathbf{p}'$. Hence, $\mathfrak{w}(e^*, 1) \geq \mathfrak{w}(e^*, \tilde{\mathbf{p}}) > \mathfrak{w}(e^*, p^*)$.[9]

If $e^* < w_*$, one has $\mathfrak{w}(e^*, 0) \geq \mathfrak{w}(e^*, \mathbf{p}')$ for all strategies $\mathbf{p}'$. Again, $\mathfrak{w}(e^*, 0) \geq \mathfrak{w}(e^*, \tilde{\mathbf{p}}) > \mathfrak{w}(e^*, p^*)$.

We have shown that if (e^*, p^*) is not an equilibrium, a stationary Markovian strategy of the agent or the principal exists which, for that player, is a profitable deviation. This is the "if" part of the Proposition.

The "only if" part is trivial. □

Proof of Lemma 1. For each n, let

$$S_n := \{a \mid p_n(e_n + a) = 1\},$$

and let $S := \bigcap_{n=1}^{\infty} S_n$. Then, for each n and all e, we have

$$\mathfrak{q}(e, p_n) = P\big(A \in (e_n - e) + S_n\big).$$

We define strategies $\tilde{p}_n$ and $\tilde{p}$ by

$$\tilde{p}_n(w) := \mathbb{1}\{w - e_n \in S\},$$

and

$$\tilde{p}(w) := \mathbb{1}\{w - \tilde{e} \in S\}.$$

By Part (i) of Proposition 2, we have $\mathfrak{q}(e_n, p_n) = 1$; hence $P(A \in S_n) = 1$ for all n, and thus $P(A \in S) = 1$. It follows that $\mathfrak{q}(e_n, \tilde{p}_n) = 1$ for all n. Since $S_n \subseteq S$, we have $\mathfrak{q}(e, \tilde{p}_n) \leq \mathfrak{q}(e, p_n)$ for all $e \in [\underline{e}; e^{\sup})$. Since, in addition, e_n is a best response to p_n, and since $\mathfrak{q}(e_n, p_n) = \mathfrak{q}(e_n, \tilde{p}_n) = 1$, we can conclude that e_n is a best response to $\tilde{p}_n$. Thus, for all n, the strategy combination $(e_n, \tilde{p}_n)$ is an equilibrium.

Suppose $(\tilde{e}, \tilde{p})$ is not an equilibrium. Then an effort level $e' \in (\underline{e}; \tilde{e})$ exists such that $\mathfrak{v}(e', \tilde{p}) > \mathfrak{v}(\tilde{e}, \tilde{p})$, which implies

$$\frac{v(e') - v_*}{1 - \beta\mathfrak{q}(e', \tilde{p})} - \frac{v(\tilde{e}) - v_*}{1 - \beta\mathfrak{q}(\tilde{e}, \tilde{p})} > 0.$$

[9] If one wants to prove the proposition for some game $\Gamma(\tau)$ with $\tau \in [-\infty; \infty]$ arbitrary (see Section 4) instead of the game $\Gamma(-\infty)$, the strategy 1 has to be replaced by the strategy $\mathbb{1}_{[\tau;\infty)}$ in this paragraph.

Let $\delta_n := \tilde{e} - e_n$. By the continuity of v and $\delta_n \to 0$, some n exists such that $e' - \delta_n > \underline{e}$ and

$$\frac{v(e' - \delta_n) - v_*}{1 - \beta\mathfrak{q}(e', \tilde{p})} - \frac{v(\tilde{e} - \delta_n) - v_*}{1 - \beta\mathfrak{q}(\tilde{e}, \tilde{p})} > 0.$$

Since $\mathfrak{q}(e', \tilde{p}) = \mathfrak{q}(e' - \delta_n, \tilde{p}_n)$, $\mathfrak{q}(\tilde{e}, \tilde{p}) = \mathfrak{q}(\tilde{e} - \delta_n, \tilde{p}_n)$, and $\tilde{e} - \delta_n = e_n$, it follows that

$$\mathfrak{v}(e' - \delta_n, \tilde{p}_n) > \mathfrak{v}(e_n, \tilde{p}_n),$$

which contradicts the fact that $(e_n, \tilde{p}_n)$ is an equilibrium. □

A.2. Proofs for Section 4

Proof of Lemma 2. Part (i), non-emptiness: Consider a threshold strategy $\mathbb{1}_{[b;\infty)}$. If $b \in \{-\infty, +\infty\}$, then $\underline{e}$ is a best response. Now suppose that $b \in (-\infty; +\infty)$. Since $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = F_{-A}(e - b)$, the function

$$[\underline{e}; \overline{e}] \to \mathbb{R}, \quad e \mapsto \mathfrak{v}(e, \mathbb{1}_{[b;\infty)})$$

is right-continuous. Further, it is bounded from above. Let

$$s := \sup\{\mathfrak{v}(e, \mathbb{1}_{[b;\infty)}) \mid e \in [\underline{e}; \overline{e}]\}.$$

A sequence (e_j) of points $e_j \in [\underline{e}; \overline{e}]$ exists such that $\mathfrak{v}(e_j, \mathbb{1}_{[b;\infty)}) \to s$ for $j \to \infty$. Since the interval $[\overline{e}; \underline{e}]$ is compact, the sequence has an accumulation point e^*. By taking a subsequence if necessary, we can assume that the sequence (e_j) converges to e^* and that it is (A) decreasing or (B) increasing. Consider Case (A). From the right-continuity of the function $e \mapsto \mathfrak{v}(e, \mathbb{1}_{[b;\infty)})$, it follows that

$$\mathfrak{v}(e^*, \mathbb{1}_{[b;\infty)}) = \lim_{j \to \infty} \mathfrak{v}(e_j, \mathbb{1}_{[b;\infty)}) = s.$$

In Case (B), the fact that the function $e \mapsto \mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = F_A(e - b)$ is monotonically increasing yields

$$\mathfrak{v}(e^*, \mathbb{1}_{[b;\infty)}) \geq \lim_{j \to \infty} \mathfrak{v}(e_j, \mathbb{1}_{[b;\infty)}) = s.$$

In either case, $\mathfrak{v}(e^*, \mathbb{1}_{[b;\infty)}) = s$; thus e^* is a best response to $\mathbb{1}_{[b;\infty)}$ and, hence, $e^* \in \mathcal{R}(b)$.

The compactness of $\mathcal{R}(b)$ is proved below.

Part (ii): The first statement, $\mathcal{R}(-\infty) = \mathcal{R}(+\infty) = \{\underline{e}\}$, is obvious. To prove that $\lim_{b\to-\infty}\big(\sup\mathcal{R}(b)\big) = \underline{e}$, let, for $b \in (-\infty; +\infty)$,

$$H(b) := \sup\Big\{e \in \big[\underline{e}; e^{\text{indiff}}\big) \;\Big|\; \mathsf{v}(e,1) \geq \mathsf{v}\big(\underline{e}, F_{-A}(\underline{e}-b)\big)\Big\}.$$

For $b \to -\infty$, we have $F_{-A}(\underline{e} - b) \to 1$; hence $H(b) \to \underline{e}$. Since $\sup\mathcal{R}(b) \leq H(b)$ for all b, the assertion follows. Now, for $b \in (-\infty; +\infty)$, let

$$\tilde{H}(b) := \sup\Big\{e \in \big[\underline{e}; e^{\text{indiff}}\big) \;\Big|\; \mathsf{v}\big(e, F_{-A}(e-b)\big) \geq \mathsf{v}(\underline{e}, 0)\Big\}.$$

Since $\sup\mathcal{R}(b) \leq \tilde{H}(b)$ for all b and since $\tilde{H}(b) \to \underline{e}$ for $b \to +\infty$, it follows that $\lim_{b\to+\infty}\big(\sup\mathcal{R}(b)\big) = \underline{e}$.

Part (iii): Since $\mathcal{G}_{\mathcal{T}}$ is the intersection of $\mathcal{G}$ and the closed set

$$\big([-\infty; +\infty] \times \{\underline{e}\} \times \{0\}\big) \cup \big([-\infty; +\infty] \times [w_*; \overline{e}] \times [0; 1]\big),$$

the compactness of $\mathcal{G}_{\mathcal{T}}$ is clear once we know that $\mathcal{G}$ is compact.

In order to demonstrate that $\mathcal{G}$ is compact, we will show that $\mathcal{G}$ is a closed subset of the set $C := [-\infty; +\infty] \times [\underline{e}; \overline{e}] \times [0; 1]$, which is compact. Consider a sequence of points $(b_n, e_n, q_n) \in \mathcal{G}$ converging to some point $(b^*, e^*, q^*) \in C$. We are going to demonstrate that $(b^*, e^*, q^*) \in \mathcal{G}$. We distinguish the cases (A) $b^* = -\infty$, (B) $b^* = +\infty$ and (C) $b^* \in (-\infty; +\infty)$.

Case (A): By Part (ii) of the proposition, we have $e^* \equiv \lim_{n\to\infty} e_n = \underline{e}$. In addition, one has $b^* \equiv \lim_{n\to\infty} b_n = -\infty$ and thus $q^* \equiv \lim_{n\to\infty} q_n = \lim_{n\to\infty} F_{-A}(e_n - b_n) = 1 = \mathfrak{q}(\underline{e}, \mathbb{1}_{[-\infty;\infty)})$. Hence, $(b^*, e^*, q^*) = (-\infty, \underline{e}, 1)$, which is contained in $\mathcal{G}$ by the definition of $\mathcal{G}$.

Case (B): As in Case (A), we have $e^* \equiv \lim_{n\to\infty} e_n = \underline{e}$ by Part (ii) of the proposition. One has $b^* \equiv \lim_{n\to\infty} b_n = +\infty$ and thus $q^* \equiv \lim_{n\to\infty} q_n = \lim_{n\to\infty} F_{-A}(e_n - b_n) = 0 = \mathfrak{q}(\underline{e}, \mathbb{1}_{[+\infty;\infty)})$. Hence, $(b^*, e^*, q^*) = (+\infty, \underline{e}, 0)$, which again is contained in $\mathcal{G}$ by the definition of $\mathcal{G}$.

Case (C): We can obviously without loss of generality assume that $b_n \in (-\infty; +\infty)$ for all n. We first show that $q^* \equiv \lim_{n\to\infty} q_n = q'$, with $q' := \mathfrak{q}(e^*, \mathbb{1}_{[b^*;\infty)})$. To reach a contradiction, suppose this is not the case. Then we can find a *convergent* subsequence

(q_{n_k}) of the sequence (q_n) such that

$$\lim_{k\to\infty} q_{n_k} \neq q'.$$

Without loss of generality (namely through replacing the sequence by the subsequence), we can assume that this subsequence is the sequence (q_n) itself, i. e. that (q_n) is convergent with

$$\lim_{n\to\infty} q_n \neq q'. \tag{27}$$

As $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = F_{-A}(e-b)$ for $b \in (-\infty;\infty)$, the function $e \mapsto \mathfrak{q}(e, \mathbb{1}_{[b;\infty)})$ is right-continuous and increasing; hence we have

$$\limsup_{n\to\infty} q_n \leq q'. \tag{28}$$

Inequalities (27) and (28) yield

$$\lim_{n\to\infty} q_n < q'.$$

Thus, $\varepsilon > 0$ exists such that for all sufficiently large n

$$\log(1-\beta q_n) > \log(1-\beta q') + \varepsilon.$$

Hence, by the continuity of the function v, some neighbourhood U of e^* exists such that for all $e \in U$ and all sufficiently large n

$$\log\bigl(v(e_n) - v_*\bigr) - \log(1-\beta q_n) < \log\bigl(v(e) - v_*\bigr) - \log(1-\beta q') - \frac{\varepsilon}{2}\,.$$

For each n, let $e'_n := \max\{\underline{e}, e^* + b_n - b^*\}$. As $q' = \mathfrak{q}(e^*, \mathbb{1}_{[b^*;\infty)}) \leq \mathfrak{q}(e'_n, \mathbb{1}_{[b_n;\infty)})$, we can deduce that for sufficiently large n

$$\begin{aligned} &\log\bigl(v(e_n) - v_*\bigr) - \log\bigl(1 - \beta\mathfrak{q}(e_n, \mathbb{1}_{[b_n;\infty)})\bigr) \\ &\qquad < \log\bigl(v(e'_n) - v_*\bigr) - \log\bigl(1 - \beta\mathfrak{q}(e'_n, \mathbb{1}_{[b_n;\infty)})\bigr) - \frac{\varepsilon}{2}\,. \end{aligned}$$

It follows that

$$\mathfrak{v}(e_n, \mathbb{1}_{[b_n;\infty)}) < \mathfrak{v}(e'_n, \mathbb{1}_{[b_n;\infty)}),$$

and hence $e_n \notin \mathcal{R}(b_n)$ for sufficiently large n, which is the desired contradiction. Thus we have proved that $q^* = q'$.

It remains to be shown that $e^* \in \mathcal{R}(b^*)$. Suppose $e^* \notin \mathcal{R}(b^*)$. Then, $\tilde{e} \in [\underline{e}; \overline{e}]$ exists such that

$$\log\big(v(e^*) - v_*\big) - \log\big(1 - \beta\mathfrak{q}(e^*, \mathbb{1}_{[b^*;\infty)})\big)$$
$$< \log\big(v(\tilde{e}) - v_*\big) - \log\big(1 - \beta\mathfrak{q}(\tilde{e}, \mathbb{1}_{[b^*;\infty)})\big).$$

Now let $\tilde{e}_n := \max\{\underline{e}, \tilde{e} + b_n - b^*\}$ and observe that $\mathfrak{q}(\tilde{e}, \mathbb{1}_{[b^*;\infty)}) \leq \mathfrak{q}(\tilde{e}_n, \mathbb{1}_{[b_n;\infty)})$. Since v is continuous and since we have just proved that $\lim_{n\to\infty} \mathfrak{q}(e_n, \mathbb{1}_{[b_n;\infty)}) = \mathfrak{q}(e^*, \mathbb{1}_{[b^*;\infty)})$, we can conclude that

$$\lim_{n\to\infty} \Big[\log\big(v(e_n) - v_*\big) - \log\big(1 - \beta\mathfrak{q}(e_n, \mathbb{1}_{[b_n;\infty)})\big)\Big]$$
$$< \lim_{n\to\infty} \Big[\log\big(v(\tilde{e}_n) - v_*\big) - \log\big(1 - \beta\mathfrak{q}(\tilde{e}_n, \mathbb{1}_{[b_n;\infty)})\big)\Big].$$

It follows that some n exists such that $e_n \notin \mathcal{R}(b_n)$, which is a contradiction.

Part (i), compactness: Consider the projection

$$h\colon [-\infty; +\infty] \times [\underline{e}; \overline{e}] \times [0; 1] \to [\underline{e}; \overline{e}], \quad (b, e, q) \mapsto e.$$

By Part (iii), the set $\mathcal{G}$ is compact. Since the function h is continuous and $\mathcal{R}(b)$ is the image of the compact set $\mathcal{G} \cap \big(\{b\} \times [\underline{e}; \overline{e}] \times [0; 1]\big)$ under h, the set $\mathcal{R}(b)$ is compact.

Part (iv): For $e \in [\underline{e}; \overline{e}]$ and $b \in (-\infty; \infty)$ let

$$M(e; b) := \log\big(v(e) - v_*\big) - \log\big(1 - \beta F_{-A}(e - b)\big)\big).$$

$M(e; b)$ is a monotonic transformation of $\mathfrak{v}(e, \mathbb{1}_{[b;\infty)})$. Consider any $e'' \in \mathcal{R}(b'')$. Then, since e'' is a best response, $M(e, b'') \leq M(e'', b'')$ for all $e \in [\underline{e}; e'']$; hence

$$M(e, b'') - M(e'', b'') \leq 0 \quad \text{for all } e \in [\underline{e}; e''].$$

Let $\delta := b'' - b'$. Note that $\delta > 0$. By the strict concavity of the function $e \mapsto \log\big(v(e) - v_*\big)$, it follows that

$$M(e - \delta, b'' - \delta) - M(e'' - \delta, b'' - \delta) < 0 \quad \text{for all } e \in [\underline{e} + \delta; e'').$$

Since $b' = b'' - \delta$, we obtain

$$M(e, b') - M(e'' - \delta, b') < 0 \quad \text{for all } e \in [\underline{e}; e'' - \delta).$$

From this we can conclude that each best response to the strategy $\mathbb{1}_{[b';\infty)}$ is larger than or equal to $e'' - (b'' - b')$. As this holds for all $e' \in \mathcal{R}(b')$ and all $e'' \in \mathcal{R}(b'')$, we have

$$\min \mathcal{R}(b') \geq \max \mathcal{R}(b'') - (b'' - b'),$$

which is the first statement we had to prove. From this it follows that for $e' \in \mathcal{R}(b')$ and $e'' \in \mathcal{R}(b'')$,

$$\begin{aligned}\mathfrak{q}(e', \mathbb{1}_{[b';\infty)}) &= F_{-A}(e' - b') \geq F_{-A}\big(\min \mathcal{R}(b') - b'\big) \\ &\geq F_{-A}\big(\max \mathcal{R}(b'') - b''\big) \geq F_{-A}(e'' - b'') = \mathfrak{q}(e'', \mathbb{1}_{[b'';\infty)}),\end{aligned}$$

which is the second statement we had to prove.

Part (v): Consider some threshold b. By Part (ii), we can without loss of generality assume that $b \in (-\infty; \infty)$. Consider any $e \in \mathcal{R}(b)$ and any $e' \in [\underline{e}; e]$. For $e' = \underline{e}$ or $e' = e$, there is nothing to prove (take $b' = -\infty$ or $b' = b$, respectively). Hence assume that $e' \in (\underline{e}; e)$. Let

$$M := \big\{b' \leq b : \mathcal{R}(b') \cap [e'; e^{\sup}) \neq \emptyset\big\}.$$

Since $b \in M$, we have $M \neq \emptyset$. Let $b_0 := \inf M$. By Part (ii), we have $b_0 > -\infty$. By Part (iii), the set $\mathcal{G} \cap \big([-\infty; b] \times [e'; \overline{e}] \times [0; 1]\big)$ is compact. Since M is the image of this set under the projection onto the first component, M is compact, and, hence, $b_0 \in M$.

For $n \in \mathbb{N}^*$, let $b_n := b_0 - 1/n$. By Part (iv) and by the definition of M, we have

$$e' > \min \mathcal{R}(b_n) \geq \max \mathcal{R}(b_0) - (b_0 - b_n) \text{ for all } n.$$

Letting $n \to \infty$, we obtain $e' \geq \max \mathcal{R}(b_0)$. This together with the definition of M and with $b_0 \in M$ yields $e' \in \mathcal{R}(b_0)$, hence we can take $b' = b_0$. □

Proof of Proposition 7. Consider a threshold τ and an equilibrium (e^*, p^*) of $\Gamma(\tau)$ that is a proper contract equilibrium. By Parts (i), (ii), and (iv) of Remark 1, we have $\mathfrak{q}(e^*, p^*) \in (0; 1)$ and $e > w_* > \underline{e}$. Since $\mathcal{R}(b) = \{\underline{e}\}$ for $b \in \{-\infty, +\infty\}$, it necessarily follows that $\tau \in (-\infty; +\infty)$.

Each threshold equilibrium (e, p) of $\Gamma(\tau)$ falls into one of the following groups:

1. $e = \underline{e}$ and $\mathfrak{q}(e, p) = 0$,
2. $e = w_*$ and $\mathfrak{q}(e, p) \in (0; 1)$,

3. $e > w_*$, $p = \mathbb{1}_{[\tau;\infty)}$, and $\mathfrak{q}(e,p) \in (0;1)$.

In the light of Proposition 6 and of Part (iv) of Remark 1, this is clear once we can show that there are no threshold equilibria $(e, \mathbb{1}_{[b;\infty)})$ with $e > w_*$ and $b > \tau$. For this purpose, consider any threshold equilibrium $(e, \mathbb{1}_{[b;\infty)})$ of $\Gamma(\tau)$ with $e > w_*$ By Part (iv) of Remark 1, we have $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) \in (0;1)$. Since the support of the noise distribution is connected, F_{-A} is strictly increasing where it takes values from $(0;1)$. Since, further, $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = F_{-A}(e-b)$ and $\mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)}) = F_{-A}(e-\tau)$, and since $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = \mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)})$ by Proposition 6, it follows that $b = \tau$.

With the help of the above classification into groups, we now prove that the threshold equilibria can be ordered in the Pareto sense.

All equilibria within the first group are utility-equivalent for both the principal and the agent; they are strictly dominated by the second group.

All equilibria within the second group are utility-equivalent for the principal; thus a Pareto-ordering is given by the utility of the agent. Consider any two threshold equilibria $(w_*, \mathbb{1}_{[b';\infty)})$ and $(w_*, \mathbb{1}_{[b'';\infty)})$ from the second group. We have $\mathfrak{q}(w_*, \mathbb{1}_{[b;\infty)}) \in (0;1)$ for $b = b', b''$. As $\mathfrak{q}(w_*, \mathbb{1}_{[b;\infty)}) = F_{-A}(w_* - b)$ and as, by the connectedness of the support, F_{-A} is strictly increasing where it takes values from $(0;1)$, it follows that $\mathfrak{v}(w_*, \mathbb{1}_{[b';\infty)}) = \mathfrak{v}(w_*, \mathbb{1}_{[b'';\infty)})$ if and only if $b' = b''$. Hence the Pareto-ordering is strict within the second group.

Now consider any equilibrium $(w_*, \mathbb{1}_{[b;\infty)})$ from the second group and any equilibrium $(e, \mathbb{1}_{[\tau;\infty)})$ from the third group. Since $b \geq \tau$ and $e > w_*$, we have $\mathfrak{q}(w_*, \mathbb{1}_{[b;\infty)}) \leq \mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)})$; hence $\mathfrak{w}(w_*, \mathbb{1}_{[b;\infty)}) < \mathfrak{w}(e, \mathbb{1}_{[\tau;\infty)})$. Since $\mathfrak{q}(w_*, \mathbb{1}_{[b;\infty)}) \leq \mathfrak{q}(w_*, \mathbb{1}_{[\tau;\infty)})$ and since e is a best response to $\mathbb{1}_{[\tau;\infty)}$, we have $\mathfrak{v}(w_*, \mathbb{1}_{[b;\infty)}) \leq \mathfrak{v}(w_*, \mathbb{1}_{[\tau;\infty)}) \leq \mathfrak{v}(e, \mathbb{1}_{[\tau;\infty)})$. Thus, the third group strictly Pareto-dominates the second group.

Since $e \in \mathcal{R}(\tau)$ for all equilibria $(e, \mathbb{1}_{[\tau;\infty)})$ from the third group, all these equilibria are utility-equivalent for the agent. Consider any two equilibria $(e', \mathbb{1}_{[\tau;\infty)})$ and $(e'', \mathbb{1}_{[\tau;\infty)})$ from the third group. Since $\mathfrak{q}(e', \mathbb{1}_{[\tau;\infty)}) < \mathfrak{q}(e'', \mathbb{1}_{[\tau;\infty)})$ for $e' < e''$, the utility of the principal is strictly increasing in the effort level; thus, a strict Pareto-ordering within the third group is given by the effort level. □

Proof of Proposition 8. Let a welfare function H, $b \in (-\infty;\infty)$, $e \in \mathcal{R}(b)$ with $e > \max\{\underline{e}, w_*\}$, and $\varepsilon > 0$ be given. By Part (iv) of Lemma 2, we can find $\tilde{\delta} > 0$ such that

$\overline{\mathcal{R}}(b') \neq \emptyset$ for $b' \in \big(b - \tilde{\delta}; b\big)$. Without loss of generality, choose $\tilde{\delta} < e - \max\{\underline{e}, w_*\}$. For all $b' \in \big(b - \tilde{\delta}; b\big)$, one has

$$\inf_{e' \in \overline{\mathcal{R}}(b')} \mathfrak{v}(e', \mathbb{1}_{[b';\infty)}) \geq \inf_{e' \in \mathcal{R}(b')} \mathfrak{v}(e', \mathbb{1}_{[b';\infty)}) \geq \mathfrak{v}(e, \mathbb{1}_{[b';\infty)}) \geq \mathfrak{v}(e, \mathbb{1}_{[b;\infty)}); \tag{29}$$

here, the second inequality follows from the fact that $\mathcal{R}(b')$ is the set of best responses, and the third inequality is due to $\mathfrak{q}(e, \mathbb{1}_{[b';\infty)}) \geq \mathfrak{q}(e, \mathbb{1}_{[b;\infty)})$.

Since the function $(e'', b'') \mapsto \mathfrak{q}(e'', \mathbb{1}_{[b'';\infty)})$ is increasing in e'' and decreasing in b'', and since $\tilde{\delta} < e - \max\{\underline{e}, w_*\}$, Part (iv) of Lemma 2 yields, for $b' \in \big(b - \tilde{\delta}; b\big)$,

$$\inf_{e' \in \overline{\mathcal{R}}(b')} \mathfrak{w}(e', \mathbb{1}_{[b';\infty)}) \geq \mathfrak{w}\big(e - (b - b'), \mathbb{1}_{[b';\infty)}\big) \geq \mathfrak{w}\big(e - (b - b'), \mathbb{1}_{[b;\infty)}\big). \tag{30}$$

Since H is continuous and increasing, and since $\mathfrak{w}$ is continuous and increasing in the first argument, we can find $\delta \in \big(0; \tilde{\delta}\big)$ such that

$$H\big(\mathfrak{v}(e, \mathbb{1}_{[b;\infty)}), \mathfrak{w}(e - (b - b'), \mathbb{1}_{[b;\infty)})\big) > \mathfrak{H}(e, \mathbb{1}_{[b;\infty)}) - \varepsilon \quad \text{for all } b' \in (b - \delta; b). \tag{31}$$

Since H is continuous and increasing, Inequalities (29), (30), and (31) yield

$$\inf_{e' \in \overline{\mathcal{R}}(b')} \mathfrak{H}(e', \mathbb{1}_{[b';\infty)}) > \mathfrak{H}(e, \mathbb{1}_{[b;\infty)}) - \varepsilon \text{ for all } b' \in (b - \delta; b).$$

□

Proof of Corollary 7. Let $(\hat{e}, \hat{p})$ be a proper contract equilibrium, and let $\varepsilon > 0$. By definition, $\hat{p}$ is a threshold strategy, hence $\hat{p} = \mathbb{1}_{[\hat{b};\infty)}$ for some $\hat{b} \in [-\infty; +\infty]$. Necessarily, $\hat{b} \in (-\infty; +\infty)$; for if $\hat{b} \in \{-\infty, +\infty\}$, then $\mathcal{R}(\hat{b}) = \{\underline{e}\}$, and thus, as $\hat{e} \in \mathcal{R}(\hat{b})$, we should have $\hat{e} = \underline{e}$, which by Parts (i) and (iii) of Remark 1 is in contradiction to $(\hat{e}, \hat{p})$ being a proper contract equilibrium.

Again by Part (i) of Remark 1, we know that $\hat{q} := \mathfrak{q}(\hat{e}, \mathbb{1}_{[\hat{b};\infty)}) \in (0; 1)$. By the definition of $\mathcal{G}_{\mathcal{T}}$, we have $(\hat{b}, \hat{e}, \hat{q}) \in \mathcal{G}_{\mathcal{T}}$. By Part (iii) of Lemma 2, the set $\mathcal{G}_{\mathcal{T}}$ is compact. Hence, $\delta > 0$ exists such that $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) \in (0; 1)$ for all pairs (b, e) with $b \in (\hat{b} - \delta; \hat{b})$ and $e \in \overline{\mathcal{R}}(b)$. In addition, according to Proposition 8, we can by choosing δ sufficiently small achieve that for all these b, one has $\overline{\mathcal{R}}(b) \neq \emptyset$ and $\mathfrak{H}(e, \mathbb{1}_{[b;\infty)}) > \mathfrak{H}(\hat{e}, \mathbb{1}_{[\hat{b};\infty)}) - \varepsilon$. By Corollary 4, we can find $b^* \in (\hat{b} - \delta; \hat{b})$ such that $\overline{\mathcal{R}}(b^*)$ contains exactly one element, which we call e^*.

Let $\tau := b^*$. By Part (i) of Proposition 6, the strategy profile $(e^*, \mathbb{1}_{[b^*;\infty)})$ is indeed an equilibrium of $\Gamma(\tau)$. By construction, it satisfies Inequality (15).

We now show that Part (iii) of the corollary is true. Consider any threshold equilibrium $(e, \mathbb{1}_{[b;\infty)})$ of $\Gamma(\tau)$ with $e > \max\{\underline{e}, w_*\}$. By construction, $\mathfrak{q}(e^*, \mathbb{1}_{[\tau;\infty)}) \in (0; 1)$. Since $\mathbb{1}_{[b;\infty)} \in \Sigma_{\mathrm{P}}(\tau)$, we necessarily have $b \geq \tau$ and thus $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) \leq \mathfrak{q}(e^*, \mathbb{1}_{[\tau;\infty)}) < 1$, by Part (iv) of Lemma 2. Since $e > w_*$, it follows that $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) \in (0; 1)$. By Part (i) of Proposition 6, we have $F_{-A}(e - b) = \mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = \mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)}) = F_{-A}(e - \tau)$. Since the support of the noise distribution is connected, F_{-A} is strictly increasing on the segment where it takes values in $(0; 1)$; it follows that $b = \tau$. Hence $e \in \overline{\mathcal{R}}(\tau)$, and thus $e = e^*$.

It remains to prove Part (ii) of the corollary. Consider any equilibrium (e, p) of $\Gamma(\tau)$ apart from (e^*, p^*). By Part (iii), $e \in \{\underline{e}, w_*\}$. As $e^* > w_* > \underline{e}$, we have $\mathfrak{w}(e, p) < \mathfrak{w}(e^*, p^*)$. Since $p \in \Sigma_{\mathrm{P}}(\tau)$ and thus $\mathfrak{q}(e, p) \leq \mathfrak{q}(e, \mathbb{1}_{[\tau,\infty)})$, and since e^* is a best response to $p^* = \mathbb{1}_{[\tau,\infty)}$, we further have

$$\mathfrak{v}(e, p) \leq \mathfrak{v}(e, \mathbb{1}_{[\tau,\infty)}) \leq \mathfrak{v}(e^*, \mathbb{1}_{[\tau,\infty)}) = \mathfrak{v}(e^*, p^*).$$

Hence (e^*, p^*) strictly dominates (e, p) in the Pareto sense. □

Proof of Proposition 9. Consider a threshold equilibrium $(\hat{e}, \hat{p})$ that maximizes the principal's utility, i. e. for which

$$\mathfrak{w}(\hat{e}, \hat{p}) = \max\big\{\mathfrak{w}(e, p) \;\big|\; (e, p) \in \mathcal{T}\big\}.$$

By definition, $\hat{p}$ is a threshold strategy; hence $\hat{p} = \mathbb{1}_{[\hat{b};\infty)}$ for some $\hat{b} \in [-\infty; +\infty]$.

Suppose $\hat{e} = w_*$. Then $e \leq w_*$ for all $(e, p) \in \mathcal{T}$; in this case there is nothing to prove.

Suppose $\hat{e} = \underline{e}$. We distinguish (i) $\underline{e} < w_*$ and (ii) $\underline{e} > w_*$: (i) If $\hat{e} = \underline{e}$ and $\underline{e} < w_*$, all threshold equilibria in $\mathcal{T}$ involve the effort level $\underline{e}$ and induce a reappointment probability of 0; hence we can choose $\hat{\tau} = -\infty$. (ii) If $\hat{e} = \underline{e}$ and $\underline{e} > w_*$, then the equilibrium $(\hat{e}, \hat{p})$ induces a reappointment probability of 1 and thus is an equilibrium of $\Gamma(-\infty)$. Again, we can choose $\hat{\tau} = -\infty$.

Now suppose $\hat{e} > \max\{\underline{e}, w_*\}$. Let $\varepsilon > 0$ be given. By setting $\mathfrak{H} = \mathfrak{w}$ (i. e. choosing $H(x, y) = y$) in Proposition 8, we can ensure the existence of some $\delta > 0$ such that $\mathfrak{w}(e, \mathbb{1}_{[b;\infty)}) > \mathfrak{w}(\hat{e}, \mathbb{1}_{[\hat{b};\infty)}) - \varepsilon$ for all pairs (b, e) with $b \in (\hat{b} - \delta; \hat{b})$, $e \in \mathcal{R}(b)$, and

$e > \max\{\underline{e}, w_*\}$. Further, the proposition tells us that such a pair (b, e) exists—we call it (b^*, e^*).

Let $\tau := b^*$. By Part (i) of Proposition 6, the strategy profile $(e^*, \mathbb{1}_{[b^*;\infty)})$ is indeed an equilibrium of $\Gamma(\tau)$. By construction, it satisfies Inequality (16).

Now consider any threshold equilibrium $(e, \mathbb{1}_{[b;\infty)})$ of $\Gamma(\tau)$ with $e > \max\{\underline{e}, w_*\}$. Since $e > \underline{e}$, necessarily $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) > 0$; hence we can distinguish the two cases (A) $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) \in (0; 1)$ and (B) $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = 1$.

Case (A): By Part (i) of Proposition 6, we have $F_{-A}(e-b) = \mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = \mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)}) = F_{-A}(e - \tau)$. Since the support of the noise distribution is connected, F_{-A} is strictly increasing on the segment where it takes values in $(0; 1)$; it follows that $b = \tau$. Hence $e \in \mathcal{R}(\tau)$, and thus $\mathfrak{w}(e, \mathbb{1}_{[b;\infty)}) = \mathfrak{w}(e, \mathbb{1}_{[\tau;\infty)}) > \mathfrak{w}(\hat{e}, \mathbb{1}_{[\hat{b};\infty)}) - \varepsilon$, by the construction of τ.

Case (B): By Part (i) of Proposition 6, we have $\mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)}) = \mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = 1$. It follows that $e \geq e^*$, as for $e < e^*$ we would have $\mathfrak{q}(e^*, \mathbb{1}_{[\tau;\infty)}) \geq \mathfrak{q}(e, \mathbb{1}_{[\tau;\infty)}) = 1$ and thus the effort level e would be a better response to $\mathbb{1}_{[\tau;\infty)}$ than e^*, in contradiction to the fact that $(e^*, \mathbb{1}_{[\tau;\infty)})$ is an equilibrium. Now $e \geq e^*$, $\mathfrak{q}(e, \mathbb{1}_{[b;\infty)}) = 1$, and the way τ was constructed give us $\mathfrak{w}(e, \mathbb{1}_{[b;\infty)}) \geq \mathfrak{w}(e^*, \mathbb{1}_{[\tau;\infty)}) > \mathfrak{w}(\hat{e}, \mathbb{1}_{[\hat{b};\infty)}) - \varepsilon$. □

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